Team Round

The Team Round consists of 15 questions. Your team has **50 minutes** to complete the Team Round. Each problem is worth 5 points.
Call a string of letters *assessable* if it is of the form '$*$ss' for some vowel $*$.
Let M be the total number of assessable strings in the problem text of all four individual rounds *in your division* combined. Submit a positive integer N; the number of bonus points your team will receive is $\lfloor 8e^{-|M-N|/12} \rfloor$.

1. Given $n \geq 1$, let A_n denote the set of the first n positive integers. We say that a bijection $f : A_n \to A_n$ has a *hump* at $m \in A_n \setminus \{1, n\}$ if $f(m) > f(m+1)$ and $f(m) > f(m-1)$. We say that f has a hump at 1 if $f(1) > f(2)$, and f has a hump at n if $f(n) > f(n-1)$. Let P_n be the probability that a bijection $f : A_n \to A_n$, when selected uniformly at random, has exactly one hump. For how many positive integers $n \leq 2020$ is P_n expressible as a unit fraction?

2. Let Γ_1 and Γ_2 be externally tangent circles with radii $\frac{1}{2}$ and $\frac{1}{8}$, respectively. The line ℓ is a common external tangent to Γ_1 and Γ_2. For $n \geq 3$, we define Γ_n as the smallest circle tangent to $\Gamma_{n-1}, \Gamma_{n-2}$, and ℓ. The radius of Γ_{10} can be expressed as $\frac{a}{b}$ where a, b are relatively prime positive integers. Find $a + b$.

3. A quadratic polynomial $f(x)$ is called *sparse* if its degree is exactly 2, if it has integer coefficients, and if there exists a nonzero polynomial $g(x)$ with integer coefficients such that $f(x)g(x)$ has degree at most 3 and $f(x)g(x)$ has at most two nonzero coefficients. Find the number of sparse quadratics whose coefficients lie between 0 and 10, inclusive.

4. Find the largest integer $x < 1000$ such that $\binom{1515}{x}$ and $\binom{1975}{x}$ are both odd.

5. Let S denote the set of all positive integers whose prime factors are elements of $\{2, 3, 5, 7, 11\}$. (We include 1 in the set S.) If
$$\sum_{q \in S} \frac{\varphi(q)}{q^2}$$
can be written as $\frac{a}{b}$ for relatively prime positive integers a and b, find $a + b$. (Here φ denotes Euler's totient function.)

6. Let $f(p)$ denote the number of ordered tuples $(x_1, x_2, \ldots, x_p)$ of nonnegative integers satisfying $\sum_{i=1}^{p} x_i = 2022$, where $x_i \equiv i \pmod{p}$ for all $1 \leq i \leq p$. Find the remainder when $\sum_{p \in \mathcal{S}} f(p)$ is divided by 1000, where $\mathcal{S}$ denotes the set of all primes less than 2022.

7. Alice, Bob, and Carol each independently roll a fair six-sided die and obtain the numbers a, b, c, respectively. They then compute the polynomial $f(x) = x^3 + px^2 + qx + r$ with roots a, b, c. If the expected value of the sum of the squares of the coefficients of $f(x)$ is $\frac{m}{n}$ for relatively prime positive integers m, n, find the remainder when $m + n$ is divided by 1000.

8. Let $\triangle ABC$ be a triangle with sidelengths $AB = 5$, $BC = 7$, and $CA = 6$. Let D, E, F be the feet of the altitudes from A, B, C, respectively. Let L, M, N be the midpoints of sides BC, CA, AB, respectively. If the area of the convex hexagon with vertices at D, E, F, L, M, N can be written as $\frac{x\sqrt{y}}{z}$ for positive integers x, y, z with $\gcd(x, z) = 1$ and y square-free, find $x + y + z$.

9. The real quartic $Px^4 + Ux^3 + Mx^2 + Ax + C$ has four different positive real roots. Find the square of the smallest real number z for which the expression $M^2 - 2UA + zPC$ is always positive, regardless of what the roots of the quartic are.

10. The sum $\sum_{k=1}^{2020} k\cos\left(\frac{4k\pi}{4041}\right)$ can be written in the form

$$\frac{a\cos(\frac{p\pi}{q})-b}{c\sin^2(\frac{p\pi}{q})},$$

where a, b, c are relatively prime positive integers and p, q are relatively prime positive integers where $p < q$. Determine $a+b+c+p+q$.

11. Let $f(z) = \frac{az+b}{cz+d}$ for $a, b, c, d \in \mathbb{C}$. Suppose that $f(1) = i$, $f(2) = i^2$, and $f(3) = i^3$. If the real part of $f(4)$ can be written as $\frac{m}{n}$ for relatively prime positive integers m, n, find m^2+n^2.

12. What is the sum of all possible $\binom{i}{j}$ subject to the restrictions that $i \geq 10, j \geq 0$, and $i+j \leq 20$? Count different i, j that yield the same value separately - for example, count both $\binom{10}{1}$ and $\binom{10}{9}$.

13. Let $\triangle TBD$ be a triangle with $TB = 6$, $BD = 8$, and $DT = 7$. Let I be the incenter of $\triangle TBD$, and let TI intersect the circumcircle of $\triangle TBD$ at $M \neq T$. Let lines TB and MD intersect at Y, and let lines TD and MB intersect at X. Let the circumcircles of $\triangle YBM$ and $\triangle XDM$ intersect at $Z \neq M$. If the area of $\triangle YBZ$ is x and the area of $\triangle XDZ$ is y, then the ratio $\frac{x}{y}$ can be expressed as $\frac{p}{q}$, where p and q are relatively prime positive integers. Find $p+q$.

14. Kelvin the frog is hopping on the coordinate plane $\mathbb{R}^2$. He starts at the origin, and every second, he hops one unit to the right, left, up, or down, such that he always remains in the first quadrant $\{(x, y) : x \geq 0, y \geq 0\}$. In how many ways can Kelvin make his first 14 jumps such that his 14th jump lands at the origin?

15. Let a_n denote the number of ternary strings of length n so that there does not exist a $k < n$ such that the first k digits of the string equals the last k digits. What is the largest integer m such that $3^m | a_{2023}$?

Team:

Write answers in table below:

Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8

Q9	Q10	Q11	Q12	Q13	Q14	Q15