P U M .: C

Team Round Solutions

We put the questions in reverse-difficulty order, and hid a message in the first letter of each problem. Happy April Fools!

1. Have $b, c \in \mathbb{R}$ satisfy $b \in (0,1)$ and $c > 0$, then let A, B denote the points of intersection of the line $y = bx + c$ with $y = |x|$, and let O denote the origin of $\mathbb{R}^2$. Let $f(b,c)$ denote the area of triangle $\triangle OAB$. Let $k_0 = \frac{1}{2022}$, and for $n \geq 1$ let $k_n = k_{n-1}^2$. If the sum $\sum_{n=1}^{\infty} f(k_n, k_{n-1})$ can be written as $\frac{p}{q}$ for relatively prime positive integers p, q, find the remainder when $p+q$ is divided by 1000.

   Proposed by Sunay Joshi

   Answer: $\boxed{484}$

   Note that the points A, B have x-coordinates $\frac{c}{-1-b} < 0$ and $\frac{c}{1-b} > 0$. Thus the area of the right triangle $\triangle OAB$ equals $f(b,c) = \frac{1}{2} \cdot \frac{c}{1+b}\sqrt{2} \cdot \frac{c}{1-b}\sqrt{2} = \frac{c^2}{1-b^2}$. As a result, the desired sum equals $\sum_{n=1}^{\infty} \frac{k^{2^n}}{1-k^{2^{n+1}}}$. We claim that this sum equals $\frac{k^2}{1-k^2}$. To see this, expand the term $\frac{k^{2^n}}{1-k^{2^{n+1}}}$ as a geometric series to find $\sum_{j=0}^{\infty} k^{j2^{n+1}+2^n}$. The exponents of this series contain all positive integers $m \equiv 2^n \pmod{2^{n+1}}$. Since the set of positive integers m such that $m \equiv 2^n \pmod{2^{n+1}}$ for some $n \geq 1$ is exactly the set of even positive integers, our sum reduces to $\sum_{\ell=1}^{\infty} k^{2\ell} = \frac{k^2}{1-k^2}$, as claimed. Plugging in $k = \frac{1}{2022}$, we find a sum of $\frac{1}{4088483}$. Thus $p + q = 4088484$ and our remainder is 484.

2. A triangle $\triangle A_0A_1A_2$ in the plane has sidelengths $A_0A_1 = 7, A_1A_2 = 8, A_2A_0 = 9$. For $i \geq 0$, given $\triangle A_iA_{i+1}A_{i+2}$, let A_{i+3} be the midpoint of A_iA_{i+1} and let G_i be the centroid of $\triangle A_iA_{i+1}A_{i+2}$. Let point G be the limit of the sequence of points $\{G_i\}_{i=0}^{\infty}$. If the distance between G and G_0 can be written as $\frac{a\sqrt{b}}{c}$, where a, b, c are positive integers such that a and c are relatively prime and b is not divisible by the square of any prime, find $a^2 + b^2 + c^2$.

   Proposed by Frank Lu

   Answer: $\boxed{422}$

   To do this, we work with vectors. Let $\vec{r_i}$ be the vector between G_i and G_{i+1}. Then, notice that, by definition, we have that $G_i = \frac{1}{3}(A_i + A_{i+1} + A_{i+2})$, meaning that $\vec{r_i} = \frac{1}{3}(A_{i+3} - A_i) = \frac{1}{6}(A_{i+1} - A_i)$. However, notice that we have that $\vec{r_i} = \frac{1}{6}(A_{i+1} - A_i) = \frac{1}{6}(\frac{1}{2}(A_{i-1} + A_{i-2}) - A_i)$ $= -\frac{1}{6}(A_i - A_{i-1}) - \frac{1}{12}(A_{i-1} - A_{i-2}) = \vec{r_{i-1}} - \frac{1}{2}\vec{r_{i-2}}$. From here, we explicitly consider one coordinate: notice then that we have the characteristic equation for, say, the $x-$coordinate, $r^2 + r + \frac{1}{2} = 0$, with the resulting solution for $x_i = Ar_1^i + Br_2^i$. But from here, notice that the solutions for r here are $\frac{-1+i}{2}$ and $\frac{-1-i}{2}$. Hence, we see that the solutions for both x, y are of this form. In particular, we see that $\vec{r_k} = \vec{a}(\frac{-1+i}{2})^k + \vec{b}(\frac{-1-i}{2})^k$. Therefore, we see that the vector between G_0 and G is equal to $\sum_{k=0}^{\infty} \vec{a}(\frac{-1+i}{2})^k + \vec{b}(\frac{-1-i}{2})^k$. But using geometric series, we see that this is just equal to $\vec{a}\frac{1}{1-\frac{-1+i}{2}} + \vec{b}\frac{1}{1-\frac{-1-i}{2}} = \vec{a}\frac{2}{3-i} + \vec{b}\frac{2}{3+i} = \vec{a}\frac{3+i}{5} + \vec{b}\frac{3-i}{5}$. We just need to find what $\vec{a}$ and $\vec{b}$ are. Returning to our original triangle, position our triangle such that $A_0 = (0,0), A_2 = (0,9)$, and A_1 has positive $y-$coordinate. Then, notice that we see that, if $A_1 = (x,y)$, we have that $x^2 + y^2 = 49, (9-x)^2 + y^2 = 64$ means that $-18x + 81 = 15$, or that $x = \frac{11}{3}$, and $y = \frac{8\sqrt{5}}{3}$. But notice then that we have that $\vec{a} + \vec{b} = \vec{r_0}$ and $\frac{-1+i}{2}\vec{a} + \frac{-1-i}{2}\vec{b} = \vec{r_1}$. Notice therefore that $\vec{a}\frac{3+i}{5} + \vec{b}\frac{3-i}{5} = 2/5\vec{r_1} + 4/5\vec{r_0}$ Simplifying this we see that this is equal to $\frac{2}{15}(A_1 - A_0 + \frac{1}{15}(A_2 - A_1) = \frac{1}{15}(A_2 + A_1 - 2A_0)$. But this is then equal to $\frac{1}{15}(\frac{38}{3}, \frac{8\sqrt{5}}{3})$. Our final answer is therefore $\frac{1}{45}\sqrt{38^2+320} = \frac{1}{45}\sqrt{1444+320} = \frac{1}{45}\sqrt{1764} = \frac{42}{45} = \frac{14}{15} = \frac{14\sqrt{1}}{15}$, or that we have $196 + 1 + 225 = 422$.

3. Provided that $\{\alpha_i\}_{i=1}^{28}$ are the 28 distinct roots of $29x^{28} + 28x^{27} + \ldots + 2x + 1 = 0$, then the absolute value of $\sum_{i=1}^{28} \frac{1}{(1-\alpha_i)^2}$ can be written as $\frac{p}{q}$ for relatively prime positive integers p, q. Find $p + q$.

   Proposed by Ben Zenker

   Answer: $\boxed{275}$

   Let $n = 30$, and let $p(x)$ denote the given polynomial. Then $\frac{1}{1-\alpha_i}$ are the roots of the function $p(\frac{x-1}{x})$. Therefore $\frac{1}{1-\alpha_i}$ are the roots of the polynomial $q(x) = x^{n-2}p(\frac{x-1}{x})$, which can be written as

$$q(x) = \sum_{k=0}^{n-2}(k+1)(x-1)^k x^{n-2-k}$$

   Let the three leading terms of $q(x)$ be denoted $ax^{n-2} + bx^{n-3} + cx^{n-4}$. By Vieta's formulas, the desired sum is given by $(-b/a)^2 - 2(c/a)$.

   We claim that the coefficient of x^{n-2-m} is given as $(-1)^m(m+1)\binom{n}{m+2}$. To see this, note that the coefficient of x^{n-2-m} in $(k+1)(x-1)^k x^{n-2-k}$ is $(k+1)(-1)^m\binom{k}{m} = (-1)^m(m+1)\binom{k+1}{m+1}$ by the Binomial Theorem. Summing over $m \le k \le n-2$, we find $(-1)^m(m+1)\binom{n}{m+2}$ by the Hockey-Stick Identity, as claimed.

   It follows that $a = \binom{n}{2}$, $b = -2\binom{n}{3}$, and $c = 3\binom{n}{4}$. Thus $b/a = \frac{-2n(n-1)(n-2)/6}{n(n-1)/2} = -\frac{2}{3}(n-2)$ and $c/a = \frac{3n(n-1)(n-2)(n-3)/24}{n(n-1)/2} = \frac{1}{4}(n-2)(n-3)$. The desired sum is therefore $\frac{4}{9}(n-2)^2 - \frac{1}{2}(n-2)(n-3)$, which reduces to $\frac{1}{18}(n-2)[(8n-16)-(9n-27)] = \frac{1}{18}(n-2)(11-n)$. Plugging in $n = 30$, the sum of squares becomes $\frac{1}{18}(28)(-19) = -\frac{266}{9}$. Thus $p = 266, q = 9$ and our answer is $266 + 9 = 275$.

4. Patty is standing on a line of planks playing a game. Define a block to be a sequence of adjacent planks, such that both ends are not adjacent to any planks. Every minute, a plank chosen uniformly at random from the block that Patty is standing on disappears, and if Patty is standing on the plank, the game is over. Otherwise, Patty moves to a plank chosen uniformly at random within the block she is in; note that she could end up at the same plank from which she started. If the line of planks begins with n planks, then for sufficiently large n, the expected number of minutes Patty lasts until the game ends (where the first plank disappears a minute after the game starts) can be written as $P(1/n)f(n) + Q(1/n)$, where P, Q are polynomials and $f(n) = \sum_{i=1}^{n}\frac{1}{i}$. Find $P(2023) + Q(2023)$.

   Proposed by Frank Lu

   Answer: $\boxed{4045}$

   Let $E(n)$ be the expected value given that the block that Masie is standing on has length n. Then, notice that if the ith plank from the left disappears, then the expected number of minutes that Masie lasts afterwards is equal to $\frac{i-1}{n}E(i)+\frac{n-i}{n}E(n-i)$; therefore, we see that we have that $E(n) = \frac{1}{n}\sum_{i=1}^{n}(\frac{i-1}{n}E(i)+\frac{n-i}{n}E(n-i)+1)$. Therefore, we see that $n^2E(n) = n^2+\sum_{j=0}^{n-1}2jE(j)$. In particular, we therefore see that $(n+1)^2E(n+1) - n^2E(n) = 2nE(n) + 2n + 1$. Now, let $F(n) = \frac{nE(n)}{n+1}$; it therefore follows that $F(n+1) - F(n) = \frac{2n+1}{(n+1)(n+2)} = \frac{3}{n+2} - \frac{1}{n+1}$. However, we also know that $E(1) = 1$, so $F(1) = \frac{1}{2}$. It therefore follows that $F(n) = \frac{1}{2} + \sum_{j=1}^{n-1}\frac{3}{n+2} - \frac{1}{n+1} =$

$\frac{1}{2}+\frac{3}{n+1}-\frac{1}{2}\sum_{j=3}^{n}\frac{2}{i}=\frac{3}{n+1}+2f(n)-3$. But this means then that $E(n)=((n+1)/n)(\frac{3}{n+1}+2f(n)-3)=(2+2/n)f(n)+3/n-3(n+1)/n=(2+2/n)f(n)-3$. But therefore we see that $P(x)=2+2x, Q(x)=-3$, and so therefore we have that our answer is $2+2\cdot 2023-3=4045$.

5. You're given the complex number $\omega=e^{2i\pi/13}+e^{10i\pi/13}+e^{16i\pi/13}+e^{24i\pi/13}$, and told it's a root of a unique monic cubic x^3+ax^2+bx+c, where a,b,c are integers. Determine the value of $a^2+b^2+c^2$.

   Proposed by Frank Lu

   Answer: $\boxed{18}$

   Observe first that the exponents of ω are precisely those of the form $2\pi ir/13$, where r is a cubic residue $\pmod{13}$. Indeed, notice that the values of r we have are $r=1, 5\equiv -8=(-2)^3 \pmod{13}, 8=2^3$, and $-1=(-1)^3$. Given as well the identity that $\sum_{j=1}^{12}e^{2\pi ij/13}=-1$, this suggests that the other two roots of this cubic are going to be the following complex numbers:

$$\omega_1=e^{4i\pi/13}+e^{20i\pi/13}+e^{32i\pi/13}+e^{48i\pi/13},$$
$$\omega_2=e^{8i\pi/13}+e^{40i\pi/13}+e^{64i\pi/13}+e^{96i\pi/13},$$

   which were obtained from ω by multiplying the cubic residues by two and four. These 12 exponents, along with 0, are $2\pi i/13$ times a complete residue class $\pmod{13}$. (This can actually be proven with Galois theory, but this is not important for the solution itself).

   We now try finding the coefficients by computing $\omega_1+\omega_2+\omega, \omega_1\omega+\omega_2\omega+\omega_1\omega_2, \omega\omega_1\omega_2$, which are obtained by Vieta's formulas. The first, as we mentioned before, is -1.

   For the other two, we analyze these terms by substituting the sums in and expanding out the products. For the second product, for instance, notice that we get $3\cdot 4\cdot 4=48$ terms. We now consider the number of these terms that are equal to $e^{2\pi ir/13}$ for each residue r. Notice that this is equal to the number of cubic residues s,t so that $s+2t\equiv r \pmod{13}$ plus the number of cubic residues s,t so that $s+4t\equiv r \pmod{13}$ plus the number where $2s+4t\equiv r \pmod{13}$. Each of these are obtained just from considering what it means for a term $e^{2\pi ir/13}$ to be obtained from one of the products.

   However, we claim that we can biject these solutions together. To see this, we can combine these equations into the form $s2^j+t2^{j+1}=r$, where s,t are cubic residues and $j\in\{0,1,2\}$. It's not hard to see then that (s,t,j) is a solution for $r=1$ if and only if $(r's, r't, j)$ is a solution if r' is a nonzero cubic residue, and that this is a bijection between solutions. If r' is twice a cubic residue, notice that (s,t,j) is a solution for $r=1$ if and only if $(r's/2, r't/2, j+1)$ is a solution if $j=0,1$, and $(4r's, 4r't, 0)$ if $j=2$. A similar procedure works for four times a cubic residue. This means that the number of times that $e^{2\pi ir/13}$ appears for each nonzero residue r is the same. And as there are four solutions for $r=1$, namely $1=(-1)+2*(1), 2*(5)+4*(1), 4*(8)+8*1, 4*(-1)+8*(-1)$, it follows there are 4 copies of each residue, which means that this pairwise product equals -4.

   Finally, we consider $\omega\omega_1\omega_2$. Notice that again, the number of terms with residue r is the number of solutions to $r=s+2t+4u$, where s,t,u are cubic residues. Here, we see again that all nonzero r have the same number of solutions. We just need to find the number of solutions to $s+2t+4u\equiv 0 \pmod{13}$. Notice that by scaling up s we may assume that $s=1$; from here notice that by going through the values for t the only solution we have are $(1,8,12)$. This means there are 4 solutions for $r=0$ and $\frac{64-4}{12}=5$ for all nonzero residues.

   Therefore, we see that the value of this product is equal to $4-5$, as the sum of this exponential for nonzero residues is equal to -1. Our polynomial is thus x^3+x^2-4x+1, and so our answer is $1+16+1=18$.

6. A sequence of integers $x_1, x_2, ...$ is *double-dipped* if $x_{n+2} = ax_{n+1} + bx_n$ for all $n \geq 1$ and some fixed integers a, b. Ri begins to form a sequence by randomly picking three integers from the set $\{1, 2, ..., 12\}$, with replacement. It is known that if Ri adds a term by picking another element at random from $\{1, 2, ..., 12\}$, there is at least a $\frac{1}{3}$ chance that his resulting four-term sequence forms the beginning of a double-dipped sequence. Given this, how many distinct three-term sequences could Ri have picked to begin with?

Proposed by Austen Mazenko

Answer: $\boxed{84}$

The main idea is that for a sequence a_1, a_2, a_3, a fourth term a_4 is double-dipped only when a_4 is a particular residue modulo $|a_2^2 - a_1a_3|$. Thus, for there to be at least 4 such values of a_4, this absolute value must equal $1, 2$, or 3; this gives casework.

If $x_2^2 \pm 1 = x_1x_3$: (double at end to reverse them) $(1,1,2)$, $(1,2,3)$, $(1,2,5)$, $(1,3,8)$, $(2,3,4)$, $(1,3,10)$, $(2,3,5)$, $(3,4,5)$, $(2,5,12)$, $(3,5,8)$, $(4,5,6)$, $(5,6,7)$, $(6,7,8)$, $(4,7,12)$, $(5,7,10)$, $(7,8,9)$, $(8,9,10)$, $(9,10,11)$, $(10,11,12)$.

If $x_2^2 \pm 2 = x_1x_3$: (double at end to reverse them) $(1,1,3)$, $(1,2,2)$, $(1,2,6)$, $(2,2,3)$, $(1,3,7)$, $(1,3,11)$, $(2,4,7)$, $(2,4,9)$, $(3,4,6)$, $(3,5,9)$, $(6,8,11)$.

If $x_2^2 \pm 3 = x_1x_3$: (double at end to reverse them) $(1,1,4)$, $(2,1,2)$, $(1,2,1)$, $(1,2,7)$, $(1,3,6)$, $(2,3,3)$, $(1,3,12)$, $(2,3,6)$, $(3,3,4)$, $(2,5,11)$, $(4,5,7)$, $(3,6,11)$, $(7,9,12)$.

In sum, we get 84 (note that $(2,1,2)$ and $(1,2,1)$ in the last case are irreversible).

7. Pick x, y, z to be real numbers satisfying $(-x+y+z)^2 - \frac{1}{3} = 4(y-z)^2$, $(x-y+z)^2 - \frac{1}{4} = 4(z-x)^2$, and $(x+y-z)^2 - \frac{1}{5} = 4(x-y)^2$. If the value of $xy + yz + zx$ can be written as $\frac{p}{q}$ for relatively prime positive integers p, q, find $p + q$.

Proposed by Sunay Joshi

Answer: $\boxed{1727}$

For convenience, let $A = \frac{1}{3}$, $B = \frac{1}{4}$, and $C = \frac{1}{5}$. Isolating the constant on the right-hand side of the first equation, we find $(-x+y+z)^2 - 4(y-z)^2 = A$. By difference of squares, this becomes $(-x+3y-z)(-x-y+3z) = A$. Consider the substitution $M = 3x - y - z$, $N = -x + 3y - z$, $P = -x - y + 3z$. Then our system reduces to $NP = A$, $MP = B$, $MN = C$. Multiplying the three together and taking the square root, we find $MNP = s\sqrt{ABC}$, where $s \in \{\pm 1\}$. Hence $M = s\sqrt{ABC}\frac{1}{A}$, $N = s\sqrt{ABC}\frac{1}{B}$, $P = s\sqrt{ABC}\frac{1}{C}$. By our definition of M, N, P, we also have $M + N + P = x + y + z$, hence $x = \frac{2M+N+P}{4} = \frac{s\sqrt{ABC}}{4}(\frac{2}{A} + \frac{1}{B} + \frac{1}{C}) = 15\frac{s\sqrt{ABC}}{4}$ and similarly $y = 16\frac{s\sqrt{ABC}}{4}$ and $z = 17\frac{s\sqrt{ABC}}{4}$. Since $s^2 = 1$, it follows that the desired quantity equals

$$xy + yz + zx = \frac{s^2 ABC}{16}(15 \cdot 16 + 16 \cdot 17 + 17 \cdot 15) = \frac{3 \cdot 16^2 - 1}{16 \cdot 60} = \frac{767}{960}$$

Hence our answer is $767 + 960 = 1727$.

8. Ryan Alweiss storms into the Fine Hall common room with a gigantic eraser and erases all integers n in the interval $[2, 728]$ such that 3^t doesn't divide $n!$, where $t = \lceil \frac{n-3}{2} \rceil$.
Find the sum of the leftover integers in that interval modulo 1000.

Proposed by Sunay Joshi

Answer: $\boxed{11}$

We claim that the sum of the integers n in the interval $[2, 3^k - 1]$ satisfying $3^t | n!$ is $\frac{1}{2}(k^2 + 5k) \cdot \frac{3^k - 1}{2} - 1$. To see this, first consider the condition $3^t | n!$. The highest power of a prime p dividing $n!$ is precisely $\nu_p(n) = \frac{n - s_p(n)}{p-1}$, where $s_p(n)$ denotes the sum of the digits of n in base p. Therefore $t \le \nu_3(n)$ is equivalent to $\lceil \frac{n-3}{2} \rceil \le \frac{n-s_3(n)}{2}$. We split into two cases based on the parity of n. For n odd, this is $\frac{n-3}{2} \le \frac{n-s_3(n)}{2}$, i.e. $s_3(n) \le 3$. For n even, this is $\frac{n-2}{2} \le \frac{n-s_3(n)}{2}$, i.e. $s_3(n) \le 2$. In the former case, it follows that the ternary representation of n must consist of either (a) one 1, (b) one 2 and one 1, or (c) three 1s. In the latter case, the ternary representation of n must consist of (d) one 2 or (e) two 1s. We now count the contribution of a given digit in the five subcases (a) through (e), where we include $n = 1$ among the valid numbers for convenience. (We will subtract $n = 1$ at the end.) One can see that the contribution is 1 for (a), $2(k-1)+(k-1) = 3(k-1)$ for (b), $\binom{k-1}{2}$ for (c), 2 for (d), and $(k-1)$ for (e). Thus each digit 3^j $(0 \le j \le k-1)$ contributes $1+3(k-1)+\binom{k-1}{2}+2+(k-1) = \frac{1}{2}(k^2+5k)$ times its value, yielding an answer of $\frac{1}{2}(k^2+5k) \cdot \frac{3^k-1}{2} - 1$, where we subtract one because we must ignore $n = 1$. Plugging in $k = 6$, we find a total of $12011 \equiv 11 \pmod{1000}$, our answer.

9. In the complex plane, let z_1, z_2, z_3 be the roots of the polynomial $p(x) = x^3 - ax^2 + bx - ab$. Find the number of integers n between 1 and 500 inclusive that are expressible as $z_1^4 + z_2^4 + z_3^4$ for some choice of positive integers a, b.

   Proposed by Sunay Joshi

   Answer: $\boxed{51}$

   For all $j \in \{1, 2, 3\}$, we have $z_j^3 = az_j^2 - bz_j + ab$. Multiplying by z_j, we find $z_j^4 = (a^2-b)z_j^2 + a^2b$. Summing over j and using the fact that $\sum z_j^2 = a^2 - 2b$, we find $\sum z_j^4 = a^4 + 2b^2$. In other words, it suffices to find the number of $n \in [1, 500]$ of the form $a^4 + 2b^2$ for $a, b \ge 1$. We first count the total number of pairs (a, b) satisfying the condition.

   $a = 1$: this implies $2b^2 \le 500 - 1^4$, hence $b \le 15$. This yields 15 solutions.

   $a = 2$: this implies $2b^2 \le 500 - 2^4$, hence $b \le 15$. This yields 15 solutions.

   $a = 3$: this implies $2b^2 \le 500 - 3^4$, hence $b \le 14$. This yields 14 solutions.

   $a = 4$: this implies $2b^2 \le 500 - 4^4$, hence $b \le 11$. This yields 11 solutions.

   Next, we eliminate duplicates. Note that if $a^4 + 2b^2 = c^4 + 2d^2$, then $a \equiv b \pmod 2$. Hence it suffices to check the cases $(a, c) = (1, 3)$ and $(a, c) = (2, 4)$.

   If $(a, c) = (1, 3)$, then $1^4 + 2b^2 = 3^4 + 2d^2$, implying $b^2 - d^2 = 40$. Thus the pair $(b - d, b + d)$ can either be $(2, 20)$ or $(4, 10)$. These yield $b = 11$ and $b = 7$ respectively, which correspond to the duplicate solutions $n = 243$ and $n = 99$.

   If $(a, c) = (2, 4)$, then $2^4 + 2b^2 = 4^4 + 2d^2$, implying $b^2 - d^2 = 24$. Thus the pair $(b - d, b + d)$ can either be $(2, 12)$ or $(4, 6)$. These yield $b = 7$ and $b = 5$ respectively, which correspond to the duplicate solutions $n = 114$ and $n = 66$.

   Subtracting the 4 duplicates from our original count of $55 = 15 + 15 + 14 + 11$, we find our answer of 51.

10. Let $\alpha, \beta, \gamma \in \mathbb{C}$ be the roots of the polynomial $x^3 - 3x^2 + 3x + 7$. For any complex number z, let $f(z)$ be defined as follows:

   $$f(z) = |z - \alpha| + |z - \beta| + |z - \gamma| - 2 \max_{w \in \{\alpha, \beta, \gamma\}} |z - w|.$$

   Let A be the area of the region bounded by the locus of all $z \in \mathbb{C}$ at which $f(z)$ attains its global minimum. Find $\lfloor A \rfloor$.

   Proposed by Oliver Thakar

   Answer: $\boxed{12}$

   The roots α, β, and γ are $-1, 2 \pm \sqrt{3}i$, which form an equilateral triangle in the complex plane. $f(z)$ is simply the sum of the smaller two of the three distances between z and the vertices of this triangle minus the largest of the distances. Ptolemy's inequality tells us that $f(z) \geq 0$ and it equals zero only when z lies on the circumcircle of the triangle with vertices α, β, γ; clearly, the circumcenter of this triangle is at $z = 1$, so the circumradius is 2. The area of the circle is $\pi \cdot 2^2$, which has floor 12.

11. For the function

$$g(a) = \max_{x \in \mathbb{R}} \left\{ \cos x + \cos\left(x + \frac{\pi}{6}\right) + \cos\left(x + \frac{\pi}{4}\right) + \cos(x+a) \right\},$$

let $b \in \mathbb{R}$ be the input that maximizes g. If $\cos^2 b = \frac{m+\sqrt{n}+\sqrt{p}-\sqrt{q}}{24}$ for positive integers m, n, p, q, find $m + n + p + q$.

Proposed by Ben Zenker

Answer: $\boxed{54}$

By the addition formula for cosine, we may rewrite $f(x)$ as

$$f(x) = (1 + \cos\frac{\pi}{6} + \cos\frac{\pi}{4} + \cos a)\cos x - (\sin\frac{\pi}{6} + \sin\frac{\pi}{4} + \sin a)\sin x = A\sin x - B\sin x$$

Factoring out $\sqrt{A^2+B^2}$, we find $f(x) = \sqrt{A^2+B^2}\cos(x-\theta)$, where $\cos\theta = \frac{A}{\sqrt{A^2+B^2}}$. It follows that $g(a) = \sqrt{A^2+B^2}$ and it suffices to maximize A^2+B^2. Expanding this expression, we find

$$g(a) = (1 + \cos\frac{\pi}{6} + \cos\frac{\pi}{4} + \cos a)^2 + (\sin\frac{\pi}{6} + \sin\frac{\pi}{4} + \sin a)^2$$
$$= (\alpha + \cos a)^2 + (\beta + \sin a)^2 = (\alpha^2 + \beta^2 + 1) + (2\alpha\cos a + 2\beta\sin a)$$
$$= (\alpha^2 + \beta^2 + 1) + 2\sqrt{\alpha^2+\beta^2}\cos(a - \varphi)$$

where $\alpha = \frac{2+\sqrt{3}+\sqrt{2}}{2}$, $\beta = \frac{1+\sqrt{2}}{2}$, and $\cos\varphi = \frac{\alpha}{\sqrt{\alpha^2+\beta^2}}$. It follows that a maximizes g iff $a = \varphi + 2\pi k$, $k \in \mathbb{Z}$, where φ is any angle satisfying $\cos\varphi = \frac{\alpha}{\sqrt{\alpha^2+\beta^2}}$. Hence the desired quantity $\cos^2 a$ equals $\cos^2\varphi$, which equals

$$\cos^2\varphi = \frac{\frac{1}{4}(2+\sqrt{2}+\sqrt{3})^2}{\frac{1}{2}(\sqrt{6}+2\sqrt{3}+3\sqrt{2}+6)} = \frac{(2+\sqrt{2}+\sqrt{3})^2}{2(2+\sqrt{2})(3+\sqrt{3})} = \frac{18+2\sqrt{3}+\sqrt{6}-3\sqrt{2}}{24}$$

Thus $m = 18, n = 12, p = 6, q = 18$ and our answer is $18 + 12 + 6 + 18 = 54$.

12. Observe the set $S = \{(x, y) \in \mathbb{Z}^2 : |x| \leq 5 \text{ and } -10 \leq y \leq 0\}$. Find the number of points P in S such that there exists a tangent line from P to the parabola $y = x^2 + 1$ that can be written in the form $y = mx + b$, where m and b are integers.

Proposed by Frank Lu

Answer: $\boxed{15}$

First, suppose that the line $y = mx+b$ is tangent to the parabola. Then, it follows that $x^2+1 = mx+b$ has exactly one solution, which in particular requires us to have that $x^2-mx+1-b = 0$ to have one solution. But from completing the square, this is only possible if $1 - b = \frac{m^2}{4}$, or that $m = 2\sqrt{1-b}$. For m, b to be integers, notice that we must have b of the form $1 - k^2$, so $m = 2k$; if m were odd, then $1 - b$, ergo b, would not be an integer.

Thus, our lines are of the form $y = 2kx + (1 - k^2)$ for some integer $k \in \mathbb{Z}$. We now seek to classify the points (x, y) that lie on a line of this form. Given such a point P in our set, we solve for k. Notice that solving for k here yields us with $k^2 - 1 - 2kx + y = 0$, or that

$k = x \pm \sqrt{x^2+1-y}$. We require this to be an integer, and we are picking x, y to also be integers. Therefore, we must have that $y = x^2+1-l^2$ for some integer l, whereby we have that $k = x \pm l$ is an integer, given x is an integer.

To count these points: notice that x^2+1 takes on the values $1, 2, 5, 10, 17, 26$, and that the negative squares are $0, -1, -4, -9, -16, -25, -36$. We now wish to count how many pairs (x, l) will yield a y that lies between -10 and 0. For $x^2+1 = 1$, these are $-1, -4, -9$, so there are 3. Repeating this procedure, we find that for $2, 5, 10, 17, 26$ that there are $2, 1, 1, 1, 1$, respectively. So the number of pairs (x, y) is thus $3 + 2 \cdot (2+1+1+1+1) = 3 + 12 = 15$.

13. Of all functions $h : \mathbb{Z}_{>0} \to \mathbb{Z}_{\geq 0}$, choose one satisfying $h(ab) = ah(b) + bh(a)$ for all $a, b \in \mathbb{Z}_{>0}$ and $h(p) = p$ for all prime numbers p. Find the sum of all positive integers $n \leq 100$ such that $h(n) = 4n$.

Proposed by Sunay Joshi

Answer: $\boxed{729}$

Setting $a = b = 1$ into the functional equation, we find $h(1) = 0 \neq 4 \cdot 1$. Thus, we may restrict our attention to $n > 1$.

We now show that if $n = \prod_{i=1}^k p_i^{e_i} > 1$, then $h(n) = (\sum_{i=1}^k e_i)n$.

To see this, we proceed by induction on $n > 1$. The base case, $n = 2$, is evident. Suppose the result holds for all numbers less than n; we show the result for n. If n is prime, then $h(n) = n$ by assumption, as desired. Otherwise, we may write the prime factorization $n = \prod_{i=1}^k p_i^{e_i}$, where $k > 1$ and $e_i > 1$ for all i. In this case, we may set $a = p_1, b = n/p_1$ into the functional equation to find

$$h(n) = p_1 h\left(\frac{n}{p_1}\right) + \frac{n}{p_1} h(p_1)$$

As $1 < \frac{n}{p_1} < n$ by assumption, we may apply the inductive hypothesis to find

$$h(n) = p_1 \cdot \left(\sum_{i=1}^k e_i - 1\right)\frac{n}{p_1} + \frac{n}{p_1} \cdot p_1 = \left(\sum_{i=1}^k e_i - 1\right)n + n = \left(\sum_{i=1}^k e_i\right)n,$$

completing the induction.

To solve $h(n) = 4n$ for $n = \prod_{i=1}^k p_i^{e_i}$, it follows that we must find all $2 \leq n \leq 100$ for which $\sum_{i=1}^k e_i = 4$. These correspond to n with the prime factorizations $\{p^4, p^3q, p^2q^2, p^2qr, pqrs\}$. Considering each of these cases in turn quickly yields the list

$$n \in \{2^4, 3^4, 2^3 \cdot 3, 2^3 \cdot 5, 2^3 \cdot 7, 2^3 \cdot 11, 3^3 \cdot 2, 2^2 \cdot 3^2, 2^2 \cdot 5^2, 2^2 \cdot 3 \cdot 5, 2^2 \cdot 3 \cdot 7, 3^2 \cdot 2 \cdot 5\}$$
$$= \{16, 81, 24, 40, 56, 88, 54, 36, 100, 60, 84, 90\},$$

with sum 729.

Remark: in number theory, the function $\frac{h(n)}{n} = \sum_i e_i$ is denoted $\Omega(n)$, and it counts the number of prime factors of n with multiplicity. Numbers with $\Omega(n) = k$ are called k-almost primes.

14. Let $\triangle ABC$ be a triangle. Let Q be a point in the interior of $\triangle ABC$, and let X, Y, Z denote the feet of the altitudes from Q to sides BC, CA, AB, respectively. Suppose that $BC = 15$, $\angle ABC = 60°$, $BZ = 8$, $ZQ = 6$, and $\angle QCA = 30°$. Let line QX intersect the circumcircle of $\triangle XYZ$ at the point $W \neq X$. If the ratio $\frac{WY}{WZ}$ can be expressed as $\frac{p}{q}$ for relatively prime positive integers p, q, find $p + q$.

Proposed by Sunay Joshi

Answer: $\boxed{11}$

Let $\theta = \angle WYZ$ and let $\varphi = \angle WZY$. By the Extended Law of Sines, $WY/WZ = \sin\varphi/\sin\theta$. Since $WYXZ$ is cyclic, $\angle WXZ = \theta$, and since $QXBZ$ is cyclic, $\angle WXZ = \angle QBZ$. Hence $\theta = \angle QBZ$. Since $\triangle QBZ$ is right with sidelengths 6, 8, 10, we have $\sin\theta = 3/5$. Similarly, since $\angle WZY = \angle WXY = \angle QCY = 30°$, $\sin\varphi = 1/2$. The desired ratio is therefore $(1/2)/(3/5) = 5/6$ and our answer is $5 + 6 = 11$.

15. Subsets S of the first 35 positive integers $\{1, 2, 3, ..., 35\}$ are called *contrived* if S has size 4 and the sum of the squares of the elements of S is divisible by 7. Find the number of contrived sets.

Proposed by Sunay Joshi

Answer: $\boxed{8605}$

There are four distinct quadratic residues modulo 7, namely $0, 1, 2, 4$, with $0^2 \equiv 0$, $1^2, 6^2 \equiv 1$, $3^2, 4^2 \equiv 2$, and $2^2, 5^2 \equiv 4$. There are five 4-tuples (a_1, a_2, a_3, a_4) with $a_1 < a_2 < a_3 < a_4$ and $a_i \in \{0, 1, 2, 4\}$ satisfying $a_1 + a_2 + a_3 + a_4 \equiv 0$, namely $(0, 0, 0, 0)$, $(0, 1, 2, 4)$, $(1, 1, 1, 4)$, $(1, 2, 2, 2)$, and $(2, 4, 4, 4)$. Among the first 35 positive integers, there are 5 numbers x with $x^2 \equiv 0$, 10 numbers with $x^2 \equiv 1$, 10 numbers with $x^2 \equiv 2$, and 10 numbers with $x^2 \equiv 4$. Thus each 4-tuple corresponds to $\binom{5}{4}$, $\binom{5}{1}\binom{10}{1}^3$, $\binom{10}{3}\binom{10}{1}$, $\binom{10}{3}\binom{10}{1}$, and $\binom{10}{3}\binom{10}{1}$ subsets, respectively. Our answer is therefore $5 + 5 \cdot 10^3 + 120 \cdot 10 + 120 \cdot 10 + 120 \cdot 10 = 8605$.