Number Theory B

1. Andrew has a four-digit number whose last digit is 2. Given that this number is divisible by 9, determine the number of possible values for this number that Andrew could have. Note that leading zeros are not allowed.

2. The smallest three positive proper divisors of an integer n are $d_1 < d_2 < d_3$ and they satisfy $d_1 + d_2 + d_3 = 57$. Find the sum of the possible values of d_2.

3. Compute the remainder when $2^{3^5} + 3^{5^2} + 5^{2^3}$ is divided by 30.

4. A *substring* of a number n is a number formed by removing some digits from the beginning and end of n (possibly a different number of digits is removed from each side). Find the sum of all prime numbers p that have the property that any substring of p is also prime.

5. Compute the number of ordered pairs of non-negative integers (x, y) which satisfy
$$x^2 + y^2 = 32045.$$

6. Let $f(n) = \sum_{\gcd(k,n)=1, 1 \le k \le n} k^3$. If the prime factorization of $f(2020)$ can be written as $p_1^{e_1} p_2^{e_2} \dots p_k^{e_k}$, find $\sum_{i=1}^{k} p_i e_i$.

7. Suppose that $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{R}$, satisfies the equation $f(x, y) = f(3x + y, 2x + 2y)$ for all $x, y \in \mathbb{Z}$. Determine the maximal number of distinct values of $f(x, y)$ for $1 \le x, y \le 100$.

8. Let $f(n) = \sum_{i=1}^{n} \frac{\gcd(i,n)}{n}$. Find the sum of all positive integers n for which $f(n) = 6$.