Number Theory B Solutions

1. The number 2021 leaves a remainder of 11 when divided by a positive integer. Find the smallest such integer.

   Proposed by: Frank Lu

   Answer: $\boxed{15}$

   Let n be our number. Then, we need our number to divide $2021 - 11 = 2010$, and to be strictly greater than 11. However, we know that $2010 = 2 \cdot 3 \cdot 5 \cdot 67$. In other words, we are looking for the smallest divisor of 2010 that is larger than 11. We see that this is 15.

2. Last year, the U.S. House of Representatives passed a bill which would make Washington, D.C. into the 51st state. Naturally, the mathematicians are upset that Congress won't prioritize mathematical interest of flag design in choosing how many U.S. states there should be. Suppose the U.S. flag must contain, as it does now, stars arranged in rows alternating between n and $n-1$ stars, starting and ending with rows of n stars, where $n \geq 2$ is some integer and the flag has more than one row. What is the minimum number of states that the U.S. would need to contain so that there are at least three different ways, excluding rotations, to arrange the stars on the flag?

   Proposed by: Ollie Thakar

   Answer: $\boxed{53}$

   We are looking for the smallest integer m expressible as $nk + (n-1)(k-1)$ for three distinct pairs (n, k), with $n, k \geq 2$, where k is the number of rows of n stars. Thus, we must have the largest value of $n \geq 4$. If we try $n = 2, 3, 4$, we note that m works if and only if it is 2 mod 3, 3 mod 5, and 4 mod 7, so the Chinese Remainder Theorem gives us that $m = 53$ is the smallest m that works. Checking $n = 5, 6$ makes clear that there is no smaller value of m works.

   Incidentally, it's actually impossible if $m = 51$, so it's a good thing that 1) the Senate is rather unproductive and 2) this is not actually a real constraint on flag design.

3. Compute the last two digits of $9^{2020} + 9^{2020^2} + \ldots + 9^{2020^{2020}}$.

   Proposed by: Nancy Xu

   Answer: $\boxed{20}$

   It is enough to compute the residue of $9^{2020} + 9^{2020^2} + \ldots + 9^{2020^{2020}}$ modulo 100. We have:

$$\begin{aligned}
9^{2020} &\equiv (10-1)^{2020} \pmod{100} \\
&\equiv \sum_{n=0}^{2020} \binom{2020}{n} (10)^n (-1)^{2020-n} \pmod{100} \\
&\equiv \binom{2020}{1} (10)(-1)^{2019} + (-1)^{2020} \pmod{100} \\
&\equiv -20200 + 1 \pmod{100} \\
&\equiv 1 \pmod{100}.
\end{aligned}$$

   Then $9^{2020^k} \equiv 1^k \pmod{100} \equiv 1 \pmod{100}$ for all k, so $9^{2020} + 9^{2020^2} + \ldots + 9^{2020^{2020}} \equiv 2020 \equiv 20 \pmod{100}$.

4. How many ordered triples of nonzero integers (a, b, c) satisfy $2abc = a + b + c + 4$?

   Proposed by: Austen Mazenko

   Answer: $\boxed{6}$

   Since $2ab - 1 \neq 0$ for integers a, b, we need $c = \frac{a+b+4}{2ab-1}$ to be an integer. If $|a|, |b| \geq 2$ then $|2ab - 1| > |a + b + 4|$ unless $a = b = 2$, so $c = \frac{8}{7}$. Thus, one of a, b is in $\{-1, 1\}$. If $a = 1$, then $(2b - 1)|(b + 5)$ and $b = 1, 6$, giving $(1, 1, 6)$ and cyclic permutations. If $a = -1$, then $(2b+1)|(b+3)$, so $b = -1$ or $b = 2$. In either case, we get $(-1, -1, 2)$ and cyclic permutations. This exhausts all possible cases, so our answer is 6.

5. Find the sum (in base 10) of the three greatest numbers less than 1000_{10} that are palindromes in both base 10 and base 5.

   Proposed by: Henry Erdman

   Answer: $\boxed{1584}$

   Noting that $2 \times 5^4 > 1000$, first we consider palindromes of the form $1XXX1_5$. Such numbers are greater than $5^4 = 625$. Note, however, that the final digit (in base 10) must be congruent to 1 modulo 5, so the greatest palindrome in both bases is of the form $6X6_{10}$. Thus we have ten options, and by trial and error, we find $676_{10} = 10201_5$ and $626_{10} = 10001_5$. These are the two largest numbers that satisfy our conditions, so we only have to find the next-largest. Note that any number greater than 4000_5 is also greater than 500_{10} and thus cannot be a palindrome in base 10 as well, since we have no number $500_{10} < x < 625_{10}$ such that the first and last digit match and are congruent to 4 modulo 5. Similarly, for $x > 3000_5$, we need $375_{10} < x < 500_{10}$ and the first and last digits of x to be congruent to 3 modulo 5. The only such palindromes are 383_{10} and 393_{10}, neither of which are palindromes in base 5. Moving down to the range $2000_5 = 250_{10} < x < 375_{10}$, $292_{10} = 2132_5$ is not a palindrome in base 5, but $282_{10} = 2112_5$ is, thus we have found our third number. Summing in base 10, $676 + 626 + 282 = 1584$.

6. Given two positive integers $a \neq b$, let $f(a, b)$ be the smallest integer that divides exactly one of a, b, but not both. Determine the number of pairs of positive integers (x, y), where $x \neq y$, $1 \leq x, y, \leq 100$ and $\gcd(f(x, y), \gcd(x, y)) = 2$.

   Proposed by: Frank Lu

   Answer: $\boxed{706}$

   First, note that $f(x, y)$ is a power of a prime; for any n that divides x but not y, if it has at least two distinct prime factors, then we can write n as $p_1^{e_1} n'$, where p_1 doesn't divide n'. Then, if $p_1^{e_1}$ divides y, then n' can't divide into y, and $n' < n$. Thus, we see that $f(x, y) = 2^e$ for some exponent $e \geq 1$. Furthermore, we see that $2|x, 2|y$ by gcd. WLOG, suppose that $f(x, y)$ divides x, but not y. Then, note that the largest power of 2 in y is $e - 1$; otherwise, either it is divisible by 2^e or that 2^{e-1} is not a divisor of y. Furthermore, the largest power of 2 dividing x is larger than that of y, giving that $e \geq 2$. Hence, $y = 2y'$, y' odd, and $x = 4x'$, x' a positive integer. Note also that either both must be divisible by 3, or neither are, else $f(x, y) \leq 3$. We will proceed with casework.

   - Case 1: x is not divisible by 3. Then, note that y' only has prime factors that are at least 5. We also know that $1 \leq y' \leq 50$, yielding $50 - \frac{50}{2} - \lfloor \frac{50}{3} \rfloor + \lfloor \frac{50}{6} \rfloor = 50 - 25 - 16 + 8 = 17$ possibilities for y'. For x', we have $25 - \lfloor \frac{25}{3} \rfloor = 25 - 8 = 17$ cases here, giving us a total of 289.
   - Case 2: x is divisible by 3. Then, $y = 6y', x = 12x'$, and all we need is that y' is odd. This yields us that we have $\lfloor \frac{100}{12} \rfloor = 8$ choices for x' and, as we need $1 \leq y' \leq 16$, 8 choices for y'. This has 64 cases.

Thus, our answer is $2 * (289 + 64) = 2 * 353 = 706$.

7. We say that a positive integer n is *divable* if there exist positive integers $1 < a < b < n$ such that, if the base-a representation of n is $\sum_{i=0}^{k_1} a_i a^i$, and the base-b representation of n is $\sum_{i=0}^{k_2} b_i b^i$, then for all positive integers $c > b$, we have that $\sum_{i=0}^{k_2} b_i c^i$ divides $\sum_{i=0}^{k_1} a_i c^i$. Find the number of non-divable n such that $1 \leq n \leq 100$.

   Proposed by: Frank Lu

   Answer: $\boxed{27}$

   First, note that if n can be written as pq, where $1 < p < q$ are positive integers, then note that the base $n-1$ representation of n is $1(n-1)+1$, and the base $q-1$ representation $p(q-1)+p$, and for $c > n-1$ we have that $((c-1)+1)|(p(c-1)+c)$. Thus, we only need to consider the positive integers that aren't primes or square of primes.

   Also, for $p > 2$, we see that base $p-1$ yields that p^2 gives $(p-1)^2 + 2(p-1) + 1$, and base $p^2 - 1$ yields $p^2 - 1 + 1$, so thus for $c \geq p^2 - 1$ we have that $(c+1)|(c^2 + 2c + 1)$.

   Now, given integer n and base-a, suppose that the base-a representation of n is $\sum_{i=0}^{k} a_i a^i$, let $p_{a,n}(x)$ be the polynomial $\sum_{i=0}^{k} a_i x^i$. Then, note that if we write $p_{a,n}(x)$ as $p_{b,n}(x)q(x)+r(x)$, where $r(x)$ has degree less than $p_{b,n}(x)$. But then note that for sufficiently large x, $p_{b,n}(x) > r(x)$.

   But then, we see that if $r(x) \neq 0$, then we see that for each integer $x > n$ that $p_{b,n}(x)|r(x)$ implies that $r(x) = 0$ for all x sufficiently large. But then r is the zero polynomial, giving that $p_{b,n}(x)|p_{a,n}(x)$.

   If $p_{b,n}(x)$ and $p_{a,n}(x)$ are the same degree, we see that the latter is a scalar multiple of the former, by say, c. But then we see that $c < p$ and we need $c|p$, contradiction.

   Otherwise, note then that if the degree of $p_{a,n}(x)$ is d, note then $1 < p_{b,n}(a) < a^d \leq p_{a,n}(a) = n$, which means that n isn't prime, contradiction.

   Thus, we see that the only non-divable numbers are primes, 4, and 1. For 4, we see the base representations 100_2 and 11_3, which is not possible.

   We thus list out the numbers: $1, 2, 3, 4, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67$ $, 71, 73, 79, 83, 89, 97$, yielding our answer of 27.

8. Find the number of ordered pairs of integers (x, y) such that 2167 divides $3x^2 + 27y^2 + 2021$ with $0 \leq x, y \leq 2166$. *Proposed by: Aleksa Milojevic*

   Answer: $\boxed{2352}$

   First, we observe that $2167 = 11 \cdot 197$, and so by Chinese Remainder Theorem we just determine the number of ways to do this for $p = 11$ and $p = 197$.

   For $p = 11$, this reduces down to the congruence $3x^2 + 27y^2 \equiv 3 \pmod{11}$, or that $x^2 + 9y^2 \equiv 1 \pmod{11}$. Since 9 is a square, we see that we can write $z = 3y$ and solve $x^2 + z^2 \equiv 1 \pmod{11}$, and get the same number of solutions (since we can then find y again given z).

   As for $p = 197$, we get that $3x^2 + 27y^2 \equiv 51 \pmod{197}$, or that $x^2 + 9y^2 \equiv 17 \pmod{197}$, which we may again write as $x^2 + z^2 \equiv 17 \pmod{197}$. Notice, however, that $197 \equiv 1 \pmod 4$, meaning that 17 is a quadratic residue of 197 if and only if 197 is one of 17, or that 10 is a square $\pmod{17}$. We can see that $10^8 \pmod{17} \equiv (-2)^4 \pmod{17} \equiv -1 \pmod{17}$, meaning that, in fact, 17 is a non-quadratic residue $\pmod{197}$.

   We now claim that the first equation has 12 solutions, and the second has 196. Here let $p = 197$ and $r = 17$. Let the number of solutions be N for $x^2 + z^2 \equiv r \pmod p$, where $r \neq 0$. Then, we have $N = \sum_{a+b=r}(1 + \left(\frac{a}{p}\right))(1 + \left(\frac{b}{p}\right))$. Thus $N = p + \sum_a \left(\frac{a}{p}\right) + \sum_b \left(\frac{b}{p}\right) + \sum_{a+b=r} \left(\frac{a}{p}\right)\left(\frac{b}{p}\right)$. The first two sums are easily seen to be 0. As for the third one, we consider the possibilities that we're allowed to have. First, suppose that a, b are both squares; notice then that, since $197 \equiv 1 \pmod 4$, -1 is a square too, so we find the number of solutions to $(x-y)(x+y) = x^2 - y^2 \equiv r \pmod{197}$. Notice that, given $x - y \neq 0$, we can find $x + y$ and thus x, y. This yields us with 196 solutions. But considering the signs that are allowed, we see that we can negate x, y freely, and since 17 isn't a square modulo 196, but -1 is, we can't have either be 0, yielding us with $\frac{p-1}{4} = 49$ solutions here.

Therefore, since we have $\frac{p+1}{2} = 99$ squares, we thus have 50 pairs where a is a square, b isn't, and so 50 where b is a square, a isn't, and therefore 48 where neither are squares. However, notice that we have two terms, namely those with $(0, 17)$ and $(17, 0)$ that we subtract because they contribute 0, not 1. But then notice that we get $49 + 48 - 50 - 50 + 2 = -1$. Therefore, we have that $N = p - 1$.

We now run this argument for $p = 11$. Notice that we end up getting that, for $a + b = 1$, since -1 is a nonquadratic residue for 11, we see that the number where a is a square, b isn't is the number of solutions $x^2 - y^2 = 1$, where $y \neq 0$. We have in total 10 solutions for x, y, of which 2 have $y = 0$, and then we divide again by 4 to get 2 solutions in total. Thus, we have $6 - 2 = 4$ pairs where both are squares, 2 again with one but not the other, and 3 where both are not squares. This then evaluates to 3. But again, here we have the pairs $(0, 1)$ and $(1, 0)$ which contribute 0 each, not 1, so we subtract 2. Therefore, we see that we have $p + 4 - 2 - 2 + 3 - 2 = p + 1$ solutions here.

Our answer is thus $12 \cdot 196 = 2352$.