



Number Theory B

1. The number 2021 leaves a remainder of 11 when divided by a positive integer. Find the smallest such integer.
2. Last year, the U.S. House of Representatives passed a bill which would make Washington, D.C. into the 51st state. Naturally, the mathematicians are upset that Congress won't prioritize mathematical interest of flag design in choosing how many U.S. states there should be. Suppose the U.S. flag must contain, as it does now, stars arranged in rows alternating between n and $n - 1$ stars, starting and ending with rows of n stars, where $n \geq 2$ is some integer and the flag has more than one row. What is the minimum number of states that the U.S. would need to contain so that there are at least three different ways, excluding rotations, to arrange the stars on the flag?
3. Compute the last two digits of $9^{2020} + 9^{2020^2} + \dots + 9^{2020^{2020}}$.
4. How many ordered triples of nonzero integers (a, b, c) satisfy $2abc = a + b + c + 4$?
5. Find the sum (in base 10) of the three greatest numbers less than 1000_{10} that are palindromes in both base 10 and base 5.
6. Given two positive integers $a \neq b$, let $f(a, b)$ be the smallest integer that divides exactly one of a, b , but not both. Determine the number of pairs of positive integers (x, y) , where $x \neq y$, $1 \leq x, y \leq 100$ and $\gcd(f(x, y), \gcd(x, y)) = 2$.
7. We say that a positive integer n is *divable* if there exist positive integers $1 < a < b < n$ such that, if the base- a representation of n is $\sum_{i=0}^{k_1} a_i a^i$, and the base- b representation of n is $\sum_{i=0}^{k_2} b_i b^i$, then for all positive integers $c > b$, we have that $\sum_{i=0}^{k_2} b_i c^i$ divides $\sum_{i=0}^{k_1} a_i c^i$. Find the number of non-divable n such that $1 \leq n \leq 100$.
8. Find the number of ordered pairs of integers (x, y) such that 2167 divides $3x^2 + 27y^2 + 2021$ with $0 \leq x, y \leq 2166$.