P U M .·. C

Number Theory A Solutions

1. Compute the remainder when $2^{3^5} + 3^{5^2} + 5^{2^3}$ is divided by 30.

   Proposed by: Matthew Kendall

   Answer: $\boxed{6}$

   Computing the remainder modulo 2:

   $$2^{3^5} + 3^{5^2} + 5^{2^3} \equiv 0 + 1^{5^2} + 1^{2^3} \equiv 0 \pmod 2,$$

   modulo 3,

   $$2^{3^5} + 3^{5^2} + 5^{2^3} \equiv (-1)^{3^5} + 0 + (-1)^{2^3} \equiv 0, \pmod 3$$

   and modulo 5 using Fermat's Little Theorem,

   $$2^{3^5} + 3^{5^2} + 5^{2^3} \equiv 2^3 + 3^1 + 0 \equiv 1 \pmod 5.$$

   By Chinese Remainder, we know the remainder must be **6**.

2. A substring of a number n is a number formed by removing any number of digits from the beginning and end of n (not necessarily the same number of digits are removed from each side). Find the sum of all prime numbers p that have the property that any substring of p is also prime.

   Proposed by: Daniel Carter

   Answer: $\boxed{576}$

   The prime numbers in question are 2, 3, 5, 7, 23, 37, 53, 73, and 373, which sum to 576. One can find the one- and two-digit primes with this property without much difficulty. Given those, the only candidate three-digit numbers are 237, 373, 537, and 737, of which only 373 is prime. Then one can see immediately that there are no four-digit primes with this property, since both the first and last three digits must also be primes with this property, i.e. they must both be 373. This also means there are no primes with five or more digits with this property.

3. Compute the number of nonnegative integral ordered pairs (x, y) such that $x^2 + y^2 = 32045$.

   Proposed by: Nancy Xu

   Answer: $\boxed{16}$

   We can write $32045 = 5 \cdot 13 \cdot 17 \cdot 29 = (1+2i)(1-2i)(2+3i)(2-3i)(1+4i)(1-4i)(2+5i)(2-5i)$, and from here we can write $x^2 + y^2 = (x - yi)(x + yi) = 32045$ by taking the product of one of each of the conjugate pairs. There are 2 options for each conjugate pair for a total of $\frac{2^4}{2} = 8$ to account for overcounting, but x and y can be swapped, so there are 16 nonnegative ordered pairs.

4. Let $f(n) = \sum_{\gcd(k,n)=1, 1 \le k \le n} k^3$. If the prime factorization of $f(2020)$ can be written as $p_1^{e_1} p_2^{e_2} \ldots p_k^{e_k}$, find $\sum_{i=1}^{k} p_i e_i$.

   Proposed by: Frank Lu

   Answer: $\boxed{818}$

   First, note that we can write $\sum_{i=1}^{n} i^3 = \sum_{d|n} \sum_{\gcd(i,n)=d} i^3 = \sum_{d|n} \sum_{\gcd(i/d,n/d)=1} d^3 i^3 = \sum_{d|n} d^3 f(n/d)$. But then we have that $(\frac{n^2+n}{2})^2 = \sum_{d|n} d^3 f(n/d)$. Now, note that, for a constant k dividing

n, we have that $\sum_{k|d, d|n} d^3 f(n/d) = \sum_{k|d, d|n} (kd')^3 f(n/d) = k^3(\frac{(n/k)^2+(n/k)}{2})^2$. Then, we can use a PIE-esque argument based on divisibility by each of the prime factors (and products of these prime factors), yielding us, after simplifying, $\frac{n^2}{4}(p_1-1)\ldots(p_k-1)(\frac{n^2}{p_1\ldots p_k}+(-1)^k)$. We thus find that $f(2020) = 2020^2/4*4*100*4039$, which equals $2^6*5^4*101^2*4039 = 2^6*5^4*7*101^2*577$, yielding us the answer of $12+20+7+202+577 = 32+786 = 818$.

5. Suppose that $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{R}$, such that $f(x, y) = f(3x + y, 2x + 2y)$. Determine the maximal number of distinct values of $f(x, y)$ for $1 \leq x, y \leq 100$.

   Proposed by: Frank Lu

   Answer: 8983

   Note that the only places where we can get distinct values for $f(x, y)$ are those that are not of the form $(3a + b, 2a + 2b)$ for some integers (a, b) in the range $1 \leq a, b \leq 100$. Observe that if $x = 3a + b, y = 2a + 2b$, then we'd have that $a = \frac{2x-y}{4}, b = \frac{3y-2x}{4}$. In other words, for this to occur, we need that $2x \equiv y, 3y \pmod 4$. But then we have that y is even and x is the same parity of $y/2$.

   Furthermore, for the points that are of the above form, in order for $1 \leq a, b \leq 100$ as well, we need $4 \leq 2x - y \leq 400$ and $4 \leq 3y - 2x \leq 400$. From here, we see that for a given value of y, we have that $y + 4 \leq 2x \leq 3y - 4$, as the other two bounds are automatically satisfied as $1 \leq x, y \leq 100$. But then with $y = 2y_1$, we see that $y_1 + 2 \leq x \leq 3y_1 - 2$. For $y_1 \leq 34$, we see that both bounds are the final bounds, meaning that, as x is the same sign as y_1, we have $y_1 - 1$ values for x. Over the values of y_1 this yields us with $33 \cdot 17 = 561$.

   For $35 \leq y_1 \leq 50$, we have $y_1 + 2 \leq x \leq 100$ as the sharp bounds. Notice that this yields us with $\lfloor \frac{100-y_1}{2} \rfloor$ values for x, again maintaining the parity condition. Summing over these values yields us with $25 + 25 + 26 + 26 + \cdots + 32 + 32 = 57 \cdot 8 = 456$ values, so in total we have $561 + 456 = 1017$ values of (x, y) that are images of the function that sends (x, y) to $(3x + y, 2x + 2y)$ within $1 \leq x, y \leq 100$.

   The number of distinct values of $f(x, y)$ is then at most $100^2 - 1017 = 8983$.

6. Let $f(n) = \sum_{i=1}^{n} \frac{\gcd(i,n)}{n}$. Find the sum of all n so that $f(n) = 6$.

   Proposed by: Frank Lu

   Answer: 1192

   Note that, the number of i so that $\gcd(i, n) = d$ is $\phi(n/d)$, if $n|d$. Then, we see that $f(n) = \sum_{i=1}^{n} \gcd(i, n) = \sum_{d|n} d\phi(n/d) = \sum_{d|n} n/d\phi(d)$ Now, suppose that n has prime factorization $n = p_1^{e_1} \ldots p_k^{e_k}$. Then, note that, since $\frac{1}{d}\phi(d)$ is multiplicative, we can write $f(n)/n$ as $\prod_{i=1}^{k} \sum_{j=0}^{e_i} \frac{1}{p_i^j}\phi(p_i^j) = \prod_{i=1}^{k}(1 + \sum_{j=1}^{e_i} \frac{1}{p_i^j} p_i^{j-1}(p - 1)) = \prod_{i=1}^{k}(1 + \sum_{j=1}^{e_i} \frac{p_i-1}{p_i}) = \prod_{i=1}^{k}(1 + \frac{e_i(p_i-1)}{p_i}). = \prod_{i=1}^{k}(\frac{(e_i+1)p_i-e_i}{p_i})$. Now, for this to be even, we need that the numerator of this product to first be even. But note that for p_i odd that $(e_i+1)p_i-e_i$ is odd, which means that one of our primes has to be 2, which say is $p_1 = 2$. Furthermore, we need that $e_1/2 + 1$ needs to be even for the product to equal 6. We thus see that $e_1 = 2, 6, 10$. For $e_1 = 10$, we see that we just have one prime factor, which means that we get the number $n = 2^{10} = 1024$. For $e_1 = 6$, we have that $e_1/2 + 1 = 4$, which is too small. However, note also that the smallest possible value for any other term in the product, with $p_i \geq 3$, is $5/3 > 3/2$. For $e_1 = 2$, we have that $e_1/2 + 1 = 2$, again is too small. We want the product of the next terms to be 3. Note that we can't have more than 2 other prime factors, the product of this is at most $5/3 \cdot 9/5 \cdot 13/7 = 39/7 > 3$. For 2 prime factors, the smallest possible value of the terms due to the other factors is $5/3 \cdot 9/3 = 3$, giving $n = 2^2 \cdot 3 \cdot 5 = 60$. For 1 prime factor, we want

$1+\frac{e_2(p_2-1)}{p_2}=3$, or that $e_2(p_2-1)=2p_2$, which requires $p_2|e_2$, or $p_2-1|2$, meaning that $p_2=3$, and that $e_2=3$. This gives $n=2^2\cdot 3^3=108$. Our total sum is thus $108+1024+60=1192$.

7. We say that a polynomial p is *respectful* if $\forall x,y\in\mathbb{Z}, y-x$ divides $p(y)-p(x)$, and $\forall x\in\mathbb{Z}, p(x)\in\mathbb{Z}$. We say that a respectful polynomial is *disguising* if it is nonzero, and all of its non-zero coefficients lie between 0 and 1, exclusive. Determine $\sum deg(f)\cdot f(2)$ over all disguising polynomials f of degree at most 5.

   Proposed by: Frank Lu

   Answer: $\boxed{290}$

   First, we claim that all respectful polynomials of degree 3 or less have integer coefficients. To see this, note that $f(0)=0$. Consider now $f(1),f(2),f(3)$. By Lagrange Interpolation, this polynomial is uniquely determined by these values. Note that we can write this polynomial as $\frac{f(3)}{6}x(x-1)(x-2)-\frac{f(2)}{2}x(x-1)(x-3)+\frac{f(1)}{2}x(x-2)(x-3)$, by the above properties. Note that the second term is a polynomial with integer coefficients. However, note that $f(3)$ is divisible by 3, and is a multiple of 2 different from $f(1)$. Hence, note that $f(3)/3$ and $f(1)$ are both integers of the same parity. Note then that this will result in an integer-coefficient polynomial, proving the desired. In particular, no disguising polynomials of degree 3 or lower exist. We now consider the case for degree at most 5 in general. For simplicity, let $a(x)=x(x-1)(x-2)\cdots(x-5)$. Again, we can write the polynomial in the above form, as $\sum_{i=1}^{5}(-1)^{5-i}\frac{1}{i!(5-i)!}\frac{a(x)}{x-i}$. Now, note that $f(5)\equiv f(2) \pmod 3$, which also means that $f(5)\equiv 10f(2) \pmod 3$. Similarly, we have that $f(4)\equiv f(1) \pmod 3$. We can thus write this expression as $x(x-1)(x-3)(x-4)(\frac{f(5)-10f(2)}{120}x-\frac{5f(5)-20f(2)}{120})+x(x-2)(x-3)(x-5)(\frac{f(4)-f(1)}{24}x-\frac{4f(4)-f(1)}{24})+\frac{f(3)}{6}x(x-1)(x-2)(x-4)(x-5)$. This shows us that the denominators of the coefficients have to divide 8; indeed, note that $f(5)-10f(2)$ and $5f(5)-20f(2)$ are both divisible by 15. Furthermore, we could alternatively re-write this by taking $\pmod 2$. This instead yields the expression (splitting the term corresponding to $i=3$ in half) $x(x-2)(x-4)(x-5)(\frac{f(1)+f(3)}{24}x+\frac{f(1)+3f(3)}{24})+x(x-1)(x-2)(x-4)(\frac{5f(3)+f(5)}{120}x+\frac{15f(3)+5f(5)}{120})+\dots$, with the remaining terms having leading coefficients $\frac{f(2)}{12}$ and $\frac{f(4)}{24}$, which have denominators that are not divisible by 4. This further shows that the denominators have to divide 4. Repeating this argument for 4th degree polynomials shows that all the denominators, in fact, have to divide 2, by only noticing the leading coefficients of the terms with $f(4),f(1)$. Checking the case for 4, notice that there can only be one such disguising polynomial; if there were two, since both of the leading coefficients are the same, it follows that their difference is somewhat disguising. But this doesn't exist for a polynomial of degree at most 3. Thus, noticing that $\frac{x^4+x^2}{2}$ is disguising, we see that this the only one for this degree. By a similar token, notice that for any two disguising polynomials of degree 5 that have the same leading coefficient, notice that $f-g$ must be an integer polynomial away from $\frac{x^4+x^2}{2}$. But the largest difference between the coefficients is -1. This means that either such polynomials are equal or differ by $\frac{x^4+x^2}{2}$, so there are at most 6 of these, with at most 2 for a given leading coefficient. For $\frac{1}{2}$ leading coefficient, we see that $\frac{x^5+x^3}{2}$ is disguising; the difference we have for x,y is $\frac{(x-y)}{2}(x^4+x^3y+x^2y^2+xy^3+y^4+x^2+xy+y^2)$. But notice that this is equivalent to $2x+2y+4xy \pmod 2$ since all powers of any integer have the same parity. Thus, we see that $\frac{x^5+x^4+x^3+x^2}{2}$ is also disguising. For the coefficient with $\frac{1}{4}$, for this to be an integer at all, this has to double to some of the disguising polynomial, meaning that this is $\frac{x^5+x^3}{4}$, possibly with some $\frac{1}{2}x^i$ terms. By the difference with $\frac{x^4+x^2}{2}$, we only need to consider 8 of these. Trying these out, note that only 4 of them actually yield integers: $\frac{x^5+2x^4+x^3}{4}$, $\frac{x^5+3x^3}{4}$, $\frac{x^5+x^3+2x}{4}$, and $\frac{x^5+2x^4+3x^3+2x}{4}$. Requiring that $f(4)$ is also divisible by 4 restricts us to the first two possibilities. The second breaks down: plugging in $x=5$ yields $\frac{5^3\cdot 28}{4}=7\cdot 125$, which is not equivalent to $1 \pmod 4$. As for the first, note that $f(3) = 27\frac{16}{4} = 27 \cdot 4 = 108$, which isn't equivalent to 1 (mod 4). This means that no other disguising polynomials exist. Our three disguising polynomials are thus $\frac{x^4+x^2}{2}$, $\frac{x^5+x^3}{2}$, and $\frac{x^5+x^4+x^3+x^2}{2}$, which take on values $10, 20$, and 30, resulting that $10 \cdot 4 + (20+30) \cdot 5 = 290$.

8. Consider the sequence given by $a_0 = 3$ and such that for $i \geq 1$, we have $a_i = 2^{a_{i-1}} + 1$. Let m be the smallest integer such that a_3^3 divides a_m. Let m' the smallest integer such that a_m^3 divides $a_{m'}$. Find the value of m'.

   Proposed by: Frank Lu

   Answer: $\boxed{35}$

   First, we show that a_i divides a_{i+1} for each nonnegative integer 1. We do this by induction. Our base case is $i = 0$, by which we see that this holds trivially. Now, say that a_i divides a_{i+1}. Then, notice that $a_{i+2} = 2^{a_{i+1}} + 1 = 2^{a_i \frac{a_{i+1}}{a_i}} + 1$. Notice that each of our a_i will be odd, meaning that we see that $a_{i+2} = (2^{a_i})^{\frac{a_{i+1}}{a_i}} + 1^{\frac{a_{i+1}}{a_i}}$ is going to be divisible by $2^{a_i} + 1 = a_{i+1}$. This finishes our induction. Now, given a prime p, let $i(p)$ be the smallest index i so that $a_{i(p)}$ is divisible by p. We claim that $v_p(a_{i(p)-1}), v_p(a_{i(p)}), v_p(a_{i(p)+1}), \dots$ is an arithmetic progression. To prove this, we again show with induction that $v_p(a_{i(p)+k}) = (k+1)v_p(a_{i(p)})$. Our base case is $k = 0$, with $k = -1$ given. From here, given this for all values before $i(p) + k$, notice that, by the lifting the exponent lemma, we have that $v_p(2^{a_{i(p)+k}} + 1) = v_p(2^{\frac{a_{i(p)+k}}{a_{i(p)+k-1}} a_{i(p)+k-1}} + 1) = v_p((2^{a_{i(p)+k-1}})^{\frac{a_{i(p)+k}}{a_{i(p)+k-1}}} + 1^{\frac{a_{i(p)+k}}{a_{i(p)+k-1}}})$, which in turn equals $v_p(a_{i(p)+k}) + v_p(\frac{a_{i(p)+k}}{a_{i(p)+k-1}}) = 2v_p(a_{i(p)+k}) - v_p(a_{i(p)+k-1}) = (k+2)a_{i(p)}$, which gives us our desired. Finally, notice that that $a_3 = 2^{513} + 1$, by trying the first two values. Notice that for each prime p that divides a_3, if j is the index so p first divides a_j, it follows that the first index k where the power is up by 3 is so that $(k + 1 - j) = 3(4 - j)$, or that $k = 12 - 3j - 1 + j = 11 - 2j$. Noticing that $a_0 = 3$, divisible by 3, we therefore have our index being $11 = m$ and therefore m', by a similar logic, equals 35.