Number Theory A

1. Compute the remainder when $2^{3^5} + 3^{5^2} + 5^{2^3}$ is divided by 30.

2. A *substring* of a number n is a number formed by removing some digits from the beginning and end of n (possibly a different number of digits is removed from each side). Find the sum of all prime numbers p that have the property that any substring of p is also prime.

3. Compute the number of ordered pairs of non-negative integers (x, y) which satisfy
$$x^2 + y^2 = 32045.$$

4. Let $f(n) = \sum_{\gcd(k,n)=1, 1 \le k \le n} k^3$. If the prime factorization of $f(2020)$ can be written as $p_1^{e_1} p_2^{e_2} \dots p_k^{e_k}$, find $\sum_{i=1}^{k} p_i e_i$.

5. Suppose that $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{R}$, satisfies the equation $f(x, y) = f(3x + y, 2x + 2y)$ for all $x, y \in \mathbb{Z}$. Determine the maximal number of distinct values of $f(x, y)$ for $1 \le x, y \le 100$.

6. Let $f(n) = \sum_{i=1}^{n} \frac{\gcd(i,n)}{n}$. Find the sum of all positive integers n for which $f(n) = 6$.

7. We say that a polynomial p is *respectful* if $\forall x, y \in \mathbb{Z}, y - x$ divides $p(y) - p(x)$, and $\forall x \in \mathbb{Z}, p(x) \in \mathbb{Z}$. We say that a respectful polynomial is *disguising* if it is nonzero, and all of its non-zero coefficients lie between 0 and 1, exclusive. Determine $\sum deg(f) \cdot f(2)$, where the sum includes all disguising polynomials f of degree at most 5.

8. Consider the sequence given by $a_0 = 3$ and such that for $i \ge 1$, we have $a_i = 2^{a_{i-1}} + 1$. Let m be the smallest integer such that a_3^3 divides a_m. Let m' the smallest integer such that a_m^3 divides $a_{m'}$. Find the value of m'.