P U M . : . C

Number Theory A

1. Compute the last two digits of $9^{2020} + 9^{2020^2} + \ldots + 9^{2020^{2020}}$.

2. How many ordered triples of nonzero integers (a, b, c) satisfy $2abc = a + b + c + 4$?

3. Find the sum (in base 10) of the three greatest numbers less than 1000_{10} that are palindromes in both base 10 and base 5.

4. Given two positive integers $a \neq b$, let $f(a, b)$ be the smallest integer that divides exactly one of a, b, but not both. Determine the number of pairs of positive integers (x, y), where $x \neq y$, $1 \leq x, y, \leq 100$ and $\gcd(f(x, y), \gcd(x, y)) = 2$.

5. We say that a positive integer n is *divable* if there exist positive integers $1 < a < b < n$ such that, if the base-a representation of n is $\sum_{i=0}^{k_1} a_i a^i$, and the base-b representation of n is $\sum_{i=0}^{k_2} b_i b^i$, then for all positive integers $c > b$, we have that $\sum_{i=0}^{k_2} b_i c^i$ divides $\sum_{i=0}^{k_1} a_i c^i$. Find the number of non-divable n such that $1 \leq n \leq 100$.

6. Find the number of ordered pairs of integers (x, y) such that 2167 divides $3x^2 + 27y^2 + 2021$ with $0 \leq x, y \leq 2166$.

7. Let $\phi(x, v)$ be the smallest positive integer n so that 2^v divides $x^n + 95$ if it exists, or 0 if no such positive integer exists. Determine $\sum_{i=0}^{255} \phi(i, 8)$.

8. What is the smallest integer a_0 such that, for every positive integer n, there exists a sequence of positive integers $a_0, a_1, \ldots, a_{n-1}, a_n$ such that the first $n-1$ are all distinct, $a_0 = a_n$, and for $0 \leq i \leq n-1$, $a_i^{a_{i+1}}$ ends in the digits $\overline{0a_i}$ when expressed without leading zeros in base 10?