Team Round Solutions

1. An evil witch is making a potion to poison the people of PUMAClandia. In order for the potion to work, the number of poison dart frogs cannot exceed 5, the number of wolves' teeth must be an even number, and the number of dragon scales has to be a multiple of 6. She can also put in any number of tiger nails. Given that the stew has exactly 2021 ingredients, in how many ways can she add ingredients for her potion to work?

   Proposed by: Nancy Xu

   Answer: $\boxed{1011}$

   It is equivalent to find the sum of the number of nonnegative multiples of 6 less than 2016, 2018, and 2020, which is $377 \times 3 = 1011$.

2. Let $k \in \mathbb{Z}_{>0}$ be the smallest positive integer with the property that $k\frac{\gcd(x,y)\gcd(y,z)}{\operatorname{lcm}(x,y^2,z)}$ is a positive integer for all values $1 \leq x \leq y \leq z \leq 121$. If k' is the number of divisors of k, find the number of divisors of k'.

   Proposed by: Frank Lu

   Answer: $\boxed{174}$

   We consider what this means with respect to a given prime power. Consider a prime power p. Notice then that, if $v_p(n)$ is the power of p in the prime factorization of n, we have that $v_p(k) + \min(v_p(x), v_p(y)) + \min(v_p(y), v_p(z)) - \max(v_p(x), 2v_p(y), v_p(z)) \geq 0$. Now, if we can force the powers of $v_p(y)$ to be maximal, and $v_p(x), v_p(z)$ to be minimal, we're done. Now, we have 30 primes less than $121 : 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97, 101, 103, 107, 109, 113$. Of these prims, notice that all but the first 5 contribute 3 divisors for each to the value of k. As for the last, notice that 2 contributes $2 \cdot 6 + 1 = 13$, 3 contributes 9, $5, 7$ contribute 5. But for 11, notice that either $v_p(y) = 1$, meaning that we get a power that is $0 + 0 - 2 = -2$, or $v_p(y) = 2$, yielding $0 + 2 - 4 = -2$; either way, we have that this contributes 3 factors. Our answer is hence $3^{26} \cdot 13 \cdot 9 \cdot 5^2 = 3^{28} \cdot 5^2 \cdot 13$ for k'. Thus, we see that the number of divisors in k' is equal to $29 \cdot 3 \cdot 2 = 174$.

3. Let $f(N) = N\left(\frac{9}{10}\right)^N$, and let $\frac{m}{n}$ denote the maximum value of $f(N)$, as N ranges over the positive integers. If m and n are relatively prime positive integers, find the remainder when $m + n$ is divided by 1000.

   Proposed by: Sunay Joshi

   Answer: $\boxed{401}$

   Note that
$$\frac{f(N+1)}{f(N)} = \frac{(N+1)\left(\frac{9}{10}\right)^{N+1}}{N\left(\frac{9}{10}\right)^N} = \frac{N+1}{N} \cdot \frac{9}{10}.$$
   If $N < 10$, then $\frac{f(N+1)}{f(N)} > 1$; if $N = 10$, then $\frac{f(N+1)}{f(N)} = 1$; and if $N > 10$, then $\frac{f(N+1)}{f(N)} < 1$. It follows that f increases until it attains its maximum at $N = 9$ and $N = 10$, and this maximum is
$$\frac{m}{n} = f(9) = f(10) = 10 \cdot \left(\frac{9}{10}\right)^{10} = \frac{9^{10}}{10^9}.$$
   Thus $m + n \equiv 9^{10} \equiv 401 \pmod{1000}$, our answer.

4. Abby and Ben have a little brother Carl who wants candy. Abby has 7 different pieces of candy and Ben has 15 different pieces of candy. Abby and Ben then decide to give Carl some

candy. As Ben wants to be a better sibling than Abby, so he decides to give two more pieces of candy to Carl than Abby does. Let N be the number of ways Abby and Ben can give Carl candy. Compute the number of positive divisors of N.

Proposed by: Nancy Xu

Answer: $\boxed{96}$

If Abby gives Carl n pieces of candy, then Ben gives Carl $n+2$ pieces of candy, so for a fix n there are $\binom{7}{n} \cdot \binom{15}{n+2}$ ways of giving candy, where $0 \le n \le 7$. Then we have:

$$\sum_{n=0}^{7} \binom{7}{n} \cdot \binom{15}{n+2} = \sum_{n=0}^{7} \binom{7}{n} \cdot \binom{15}{13-n} = \sum_{n=0}^{13} \binom{7}{n} \cdot \binom{15}{13-n}.$$

Applying Vandermonde's identity gives us $\sum_{n=0}^{13} \binom{7}{n} \cdot \binom{15}{13-n} = \binom{22}{13}$, where $\binom{22}{13} = 2^2 \cdot 5 \cdot 7 \cdot 11 \cdot 17 \cdot 19$ and has $3 \cdot 2^5 = \boxed{96}$ divisors.

5. Given a real number t with $0 < t < 1$, define the real-valued function $f(t,\theta) = \sum_{n=-\infty}^{\infty} t^{|n|}\omega^n$, where $\omega = e^{i\theta} = \cos\theta + i\sin\theta$. For $\theta \in [0, 2\pi)$, the polar curve $r(\theta) = f(t,\theta)$ traces out an ellipse E_t with a horizontal major axis whose left focus is at the origin. Let $A(t)$ be the area of the ellipse E_t. Let $A(\frac{1}{2}) = \frac{a\pi}{b}$, where a, b are relatively prime positive integers. Find $100a + b$.

   Proposed by: Sunay Joshi

   Answer: $\boxed{503}$

   Note that $f(t,\theta)$ can be written as the sum of two geometric series:

$$f(t,\theta) = \sum_{n\ge 0} t^n\omega^n + \sum_{n\ge 0} t^n\omega^{-n} - 1.$$

   Using the formula for the sum of a geometric series, we find

$$f(t,\theta) = \frac{1-t^2}{1+t^2-2t\cos\theta} = \frac{\frac{1-t^2}{1+t^2}}{1-\frac{2t}{1+t^2}\cos\theta}.$$

   The polar formula for an ellipse whose left focus is at the origin and whose major axis is along the x-axis is given by $r(\theta) = \frac{a(1-e^2)}{1-e\cos\theta}$, where $e = \frac{\sqrt{a^2-b^2}}{a}$, where a and b are the semimajor and semiminor axes, respectively. Matching parameters to $r(\theta) = f(t,\theta)$, we find $e = \frac{2t}{1+t^2}$ and $a(1-e^2) = \frac{1-t^2}{1+t^2}$. Solving for a, b, we find $a = \frac{1+t^2}{1-t^2}$ and $b = 1$.

   Thus the area is given by $A(t) = \pi ab = \pi\frac{1+t^2}{1-t^2}$. At $t = \frac{1}{2}$, we have $A(\frac{1}{2}) = \pi \cdot \frac{5/4}{3/4} = \frac{5\pi}{3}$. Thus our answer is $500 + 3 = 503$.

6. Jack plays a game in which he first rolls a fair six-sided die and gets some number n; then, he flips a coin until he flips n heads in a row and wins, or he flips n tails in a row in which case he rerolls the die and tries again. What is the expected number of times Jack must flip the coin before he wins the game?

   Proposed by: Austen Mazenko

   Answer: $\boxed{40}$

   Let the expected number be E. After a given roll of the die, by symmetry the probability that a win occurs compared to a reroll is $\frac{1}{2}$. Now, given the roll was an i, consider the expected number $E(i)$ of coin flips before a run of i occurs given that you've already flipped the coin once. Evidently, $E(1) = 0$. Then, $E(2) = \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot E(2) \implies E(2) = 1$. Next,

$E(3) = \frac{1}{4} \cdot 2 + \frac{1}{4} \cdot (2 + E(3)) + \frac{1}{2} \cdot (1 + E(3)) \implies E(3) = 6$. We have $E(4) = \frac{1}{8} \cdot 3 + \frac{1}{8} \cdot (3 + E(4)) + \frac{1}{4} \cdot (2 + E(4)) + \frac{1}{2} \cdot (1 + E(4)) \implies E(4) = 14$. Similarly, $E(5) = 30$ and $E(6) = 62$. In general, the expected number of coin flips before a run of i appears is $2^i - 1$. Thus, considering the six possible starting cases, we have

$$E = \sum_{i=1}^{6} \tfrac{1}{6} \cdot (2^i - 1 + \tfrac{E}{2}) \implies E = 40.$$

7. The roots of the polynomial $f(x) = x^8 + x^7 - x^5 - x^4 - x^3 + x + 1$ are all roots of unity. We say that a real number $r \in [0, 1)$ is *nice* if $e^{2i\pi r} = \cos 2\pi r + i \sin 2\pi r$ is a root of the polynomial f and if $e^{2i\pi r}$ has positive imaginary part. Let S be the sum of the values of nice real numbers r. If $S = \frac{p}{q}$ for relatively prime positive integers p, q, find $p + q$.

   Proposed by: Frank Lu

   Answer: $\boxed{31}$

   We try multiplying this polynomial by $x^2 - x + 1$, in an attempt to find which polynomial of the form $x^n - 1$ it divides. We see that this equals $x^{10} - x^5 + 1$, meaning that the roots of this polynomial are all 30th roots of unity. We now need to determine which of these divide into the polynomial given.

   Notice that we have the equality $(x^{15} + 1) = (x^5 + 1)(x^{10} - x^5 + 1)$. Thus, we have the roots of $x^{15} + 1$, minus those that are roots of either $x^5 + 1$ or $x^2 - x + 1$. But notice that the former are roots of the form $e^{i\pi k/5}$, where k is odd, and for the latter they have the form $e^{i\pi k/3}$, where k is odd.

   This means that the possible values of r that we have (which we note are of the form $\frac{k}{30}$ for some odd k) are $\frac{1}{30}, \frac{7}{30}, \frac{11}{30}, \frac{13}{30}, \frac{17}{30}, \frac{19}{30}, \frac{23}{30}, \frac{29}{30}$. Of these, only the first four have positive imaginary part (as positive imaginary part holds if and only if r lies between 0 and $\frac{1}{2}$). Our sum of the first four is thus $\frac{32}{30} = \frac{16}{15}$, so our answer is 31.

8. The new PUMaC tournament hosts 2020 students, numbered by the following set of labels $1, 2, \ldots, 2020$. The students are initially divided up into 20 groups of 101, with each division into groups equally likely. In each of the groups, the contestant with the lowest label wins, and the winners advance to the second round. Out of these 20 students, we chose the champion uniformly at random. If the expected value of champion's number can be written as $\frac{a}{b}$, where a, b are relatively prime integers, determine $a + b$.

   Proposed by: Frank Lu

   Answer: $\boxed{2123}$

   First, let S be the sum of the winners. We want to find $\frac{1}{20}\mathbb{E}(S)$. To find S, we write this as $\sum_{j=1}^{2020} iX_i$, where X_i is equal to 1 if and only if contestant i was the winner of their group, and zero otherwise. Now, note that $\mathbb{E}(iX_i)$ is equal to i times the probability that X_i is equal to 1. But $\mathbb{P}(X_i = 1)$ is the probability that player i is the lowest in their group. There are $\binom{2020-i}{100}$ possibilities for this, and a total of $\binom{2019}{100}$ in total for the group, meaning that our probability is just $\frac{\binom{2020-i}{100}}{\binom{2019}{100}}$. By linearity of expectation, we find that $\mathbb{E}(S) = \sum_{i=1}^{2020} \frac{i\binom{2020-i}{100}}{\binom{2019}{100}}$. We can in turn rewrite this sum as equalling $\mathbb{E}(S) = \sum_{i=1}^{2020} \sum_{j=1}^{i} \frac{\binom{2020-i}{100}}{\binom{2019}{100}}$. By changing the order of summation and using hockey-stick identity, this is equal to $\sum_{j=1}^{2020} \sum_{i=j}^{2020} \frac{\binom{2020-i}{100}}{\binom{2019}{100}} = \sum_{j=1}^{2020} \frac{\binom{2021-j}{101}}{\binom{2019}{100}}$, which in turn is $\frac{\binom{2021}{102}}{\binom{2019}{100}}$ with another application of hockey-stick. We thus see that $\mathbb{E}(S) = \frac{2021 \cdot 2020}{102 \cdot 101}$, which means that our desired expected value is equal to $\frac{2021 \cdot 2020}{102 \cdot 101 \cdot 20}$, giving us our total expected value of $\frac{2021}{102}$, which we can see is relatively prime as 2021 is not divisible by any of $2, 3, 17$ and thus yielding the answer of 2123.

9. Let AX be a diameter of a circle Ω with radius 10, and suppose that C lies on Ω so that $AC = 16$. Let D be the other point on Ω so $CX = CD$. From here, define D' to be the reflection of D across the midpoint of AC, and X' to be the reflection of X across the midpoint of CD. If the area of triangle $CD'X'$ can be written as $\frac{p}{q}$, where p, q are relatively prime, find $p + q$.

   Proposed by: Frank Lu

   Answer: $\boxed{1367}$

   First, we consider the triangle ADC. It is well-known that the reflection of the orthocenter of this triangle, reflected across the midpoint of DC, is the diametrically opposite point on this circle. In particular, notice that X' is the orthocenter of ADC. But this means that we know that D is the orthocenter of $AX'C$. This implies that $X'D'$ is a diameter of the circumcircle of $AX'C$. In particular, $CD'X'$ is a right triangle. Now, observe that, since $CX = CD$, we have that $\angle CAD = \angle CAX$. But $\sin \angle CAX = \frac{3}{5}$. Therefore, we see that $\sin \angle CAD = \frac{3}{5}$. However, we also know that, since AX is a diameter, we have that $CX = 12$, and so hence that $CD = 12$. But notice that this is less than $10\sqrt{2}$, meaning that D lies on the same semicircle defined by AX. In particular, this means that $\angle CDA$ is obtuse. Therefore, it follows that, by the Law of Sines, we have that $\sin \angle CDA = \sin \angle CAD \frac{AC}{CD} = \frac{4}{5}$, and so $\cos \angle CDA = -\frac{3}{5}$. But it is well-known property of orthocenters that $\angle CX'A = 180 - \angle CDA$. Similarly, we see that $\angle X'CA = 90 - \angle DX'C = 90 - \angle DAC$. Therefore, we have that $\cos \angle CX'A = \frac{3}{5}$ and $\cos \angle X'CA = \frac{3}{5}$, meaning that this triangle is isosceles, with $AC = AX'$. Hence, as $AC = 16$, we have that $CX' = 216\frac{3}{5} = \frac{96}{5}$. But notice then that $X'D' = \frac{16}{\frac{4}{5}} = 20$, meaning that we have that $D'X' = \frac{28}{5}$, and so the area is $\frac{96 \cdot 28}{2 \cdot 25} = \frac{96 \cdot 14}{25} = \frac{1344}{25}$, and so $p + q = 1367$.

10. Determine the number of pairs (a, b), where $1 \leq a \leq b \leq 100$ are positive integers, so that $\frac{a^3+b^3}{a^2+b^2}$ is an integer.

    Proposed by: Frank Lu

    Answer: $\boxed{122}$

    Factor $a = dx, b = dy$, where x, y are relatively prime (so $d = \gcd(a, b)$.) Now, substituting these into the equation, we get $\frac{a^3+b^3}{a^2+b^2} = d\frac{x^3+y^3}{x^2+y^2}$. We now consider x^3+y^3 and x^2+y^2. Notice that the greatest common divisor of these two expressions is equal to that of $y^3 - xy^2 = y^2(y-x)$ and x^2+y^2. However, since x, y are relatively prime, y^2 is relatively prime to x^2+y^2. Furthermore, we see that $y - x, x^2 + y^2$ have the same gcd as $y - x$ and $2yx$. Therefore, if $x - y$ is odd, we see that these two are relatively prime; otherwise they only share a common factor of 2.

    Therefore, we see that d must be divisible by $\frac{x^2+y^2}{2}$ if $x^2 + y^2$ is even, and $x^2 + y^2$ if this is odd. We can then go through the various cases that we can have for x, y; notice that both of them must be at most 9. In running through this list, we need to have $y \geq x$, and case work through the value of x. Note that $x \leq 6$. In fact, noticing that b must be divisible by $y(x^2 + y^2)$, we need $y \leq 4$.

    For $x = 1$, we see that the smallest value of b is $y(y^2 + 1)$ if y is even and $\frac{y(y^2+1)}{2}$ if y is odd, and all values of b are divisible by this smallest value. We thus need $y \leq 5$. $y = 1$ yields us with 100 pairs, $y = 2$ yields us with $\frac{100}{10} = 10$, $y = 3$ yields us with $\lfloor \frac{100}{15} \rfloor = 6$, $y = 4$ only gives one pair, and $y = 5$ yields us with exactly one pair too.

    For $x = 2$, we only have the pair $(2, 3)$ for (x, y), with the smallest value of b being $(3(2^2+3^2)) = 3 \cdot 13 = 39$, which only gives us 2 pairs. Finally, for $x = 3$, we have the pairs $4(3^2 + 4^2) = 100$ and $\frac{5(3^2+5^2)}{2} = 85$.

Our total is $100 + 10 + 6 + 1 + 1 + 2 + 2 = 122$.

11. ABC is a triangle where $AB = 10$, $BC = 14$, and $AC = 16$. Let DEF be the triangle so that DE is parallel to AB, EF is parallel to BC, DF is parallel to AC, the circumcircle of ABC is DEF's inscribed circle, and D, A are on the same side of BC. Line EB meets the circumcircle of ABC again at a point X. Find BX^2.

Proposed by: Frank Lu

Answer: $\boxed{196}$

First, observe that by our parallel lines, we have that DEF is homothetic to triangle ABC. Let P be the center of this homothety. In addition, let I be the incenter of ABC and O be the circumcenter of ABC. Notice in particular that the homothety that sends triangle ABC to triangle DEF also sends the incenter of ABC to the circumcenter of ABC. Notice, however, that the semiperimeter of our triangle is equal to 20, yielding us an area for the triangle as $\sqrt{20 \cdot 10 \cdot 6 \cdot 4} = 40\sqrt{3}$. Therefore, we find that the inradius of triangle ABC is equal to $2\sqrt{3}$ and the circumradius of triangle ABC is equal to $\frac{10 \cdot 14 \cdot 16}{4 \cdot 40\sqrt{3}} = \frac{14\sqrt{3}}{3}$. In particular, we see that the ratio of our homothety is equal to $\frac{7}{3}$, with the order of points being P, I, O by the condition that A, D are on the same side of BC.

From here, if I' is the point at which the incircle of ABC is tangent to AB, and I'' the point where the circumcircle of ABC is tangent to DE, we see that, first, that $BI' = \frac{10+14-16}{2} = 4$, and so therefore that $EI'' = \frac{28}{3}$.

Now, by our homothety, notice that E, B, P are collinear, with $EP/BP = 7/3$. We now wish to find what BP is equal to. To find this, drop the perpendiculars from P and O onto AB, meeting AB at points P' and O', respectively. We already know that $BO' = 5$ and $BI' = 4$, meaning that by our homothety we have that BP' is so that $BI' = \frac{4}{7}BP' + \frac{3}{7}BO'$, or that $BP' = \frac{13}{4}$. Similarly, we see that, since the circumradius of ABC is equal to $\frac{14\sqrt{3}}{3}$, we have that $OO' = \sqrt{\frac{196}{3} - 25} = \sqrt{\frac{121}{3}} = \frac{11\sqrt{3}}{3}$ and $II' = 2\sqrt{3}$, meaning that again we have that PP' is so that $II' = \frac{4}{7}PP' + \frac{3}{7}OO'$, or that $PP' = \frac{3\sqrt{3}}{4}$. This means that BP, by the Pythagorean theorem, is equal to $\frac{1}{4}\sqrt{169 + 27} = \frac{7}{2}$, and so hence that BE, by homothety, is equal to $\frac{14}{3}$. However, by power of a point, we also know that $BE \cdot EX = (EI'')^2 = \frac{784}{9}$, meaning that $EX = \frac{784}{9}\frac{3}{14} = \frac{56}{3}$, and so hence that $BX = \frac{56}{3} - \frac{14}{3} = 14$, and so $BX^2 = 196$.

12. Given an integer a_0, we define a sequence of real numbers $a_0, a_1, \ldots$ using the relation

$$a_i^2 = 1 + ia_{i-1}^2,$$

for $i \geq 1$. An index j is called *good* if a_j can be an integer for some a_0. Determine the sum of the indices j which lie in the interval $[0, 99]$ and which are not good.

Proposed by: Frank Lu

Answer: $\boxed{4946}$

We claim that the only ones that are possible, where a_j can be an integer, are $j = 0, 1, 3$. To see this, we first claim that, given a_0, we have that $a_k^2 = \sum_{j=0}^{k} j!\binom{k}{j} + k!a_0^2$, where $k \geq 1$. To prove this, we use induction: for $j = 1$, this is given by the formula. From here, we notice that for a_{k+1}^2, this is equal to $1 + (k+1)a_k^2 = 1 + \sum_{j=1}^{k+1}(j+1)!\binom{k+1}{j+1} + (k+1)!a_0^2$, which gives us our desired. It's also not too hard to check that $a_2^2 = 3 + 2a_0^2$ cannot be a perfect square, by considering this equation $\pmod 9$.

Now notice that if $k \geq 3$, taking the terms modulo 3 yields us with $a_k^2 \equiv 1 + k^2 \pmod 3$, which is only a quadratic residue when k is divisible by 3. By a similar logic, notice that this is equivalent to $5 \pmod{k-2}$; for a_k^2 to be a perfect square, we need 5 to be a quadratic residue modulo $k-2$.

However, we know that, by quadratic reciprocity, we know that for each prime p dividing $k-2$, since $5 \equiv 1 \pmod 4$, we have that this is a valid quadratic residue if and only if every odd prime p dividing $k-2$ is a quadratic residue $\pmod 5$, which in turn holds if and only if every odd prime dividing $k-2$ is either 1 or 4 $\pmod 5$. Therefore, listing the odd primes that are $1, 4 \pmod 5$, we note that these are $11, 19, 29, 31, 41, 59, 61, 71, 79, 89$ (or $k-2 = 1$.) Therefore, the possible values of $k-2$ are these values, with some additional powers of 2, so $k-2$ can be either a power of 2, or one of $11, 22, 44, 88, 19, 38, 76, 29, 58, 31, 62, 41, 82, 59, 61, 71, 79, 89$. However, $k-2$ cannot be divisible by 8, since 5 is not a quadratic residue modulo 8. So this narrows our values down to $1, 2, 4, 11, 22, 44, 19, 38, 76, 29, 58, 31, 62, 41, 82, 59, 61, 71, 79, 89$ for $k-2$, or that k is one of $3, 4, 6, 13, 24, 46, 21, 40, 78, 31, 60, 33, 64, 43, 84, 61, 63, 73, 81, 91$.

From here, we can also check this $\pmod 8$, by considering the terms $j = 0, 1, 2, 3$; we get that $a_k^2 \equiv 1 + k + k(k-1) + k(k-1)(k-2) = k((k-1)^2 + 1) + 1 \pmod 8$. This has to be equivalent to either $0, 1$, or $4 \pmod 8$. Taking a look at the terms, we thus see that k has to either be divisible by 4, or it has to be equivalent to $3 \pmod 4$. So k is narrowed down to $3, 4, 24, 40, 31, 60, 64, 43, 84, 63, 91$.

Next, taking this modulo $k-1$ yields that 2 must be a quadratic residue $\pmod{k-1}$, which means that the only odd primes dividing $k-1$ are $1, 7 \pmod 8$. Checking through our list, we rule our $4, 40, 31, 60, 64, 43, 84, 91$, so we are left with just $3, 24, 63$.

Finally, we want to rule out 24 and 63. For these values, observe that modulo $k-4$ we have $a_k^2 \equiv 1 + 4 + 12 + 24 + 24 \equiv 65 \pmod{k-4}$ for $k \geq 5$. But for $k = 63$, this implies that 6 is a square modulo 59. But 2 isn't a square modulo 59 as $59 \equiv 3 \pmod 8$, but 3 is since $59 \equiv 2 \pmod 3$, and $59 \equiv 3 \pmod 4$ and employing quadratic reciprocity.

Similarly, modulo $k-5$ yields that $a_k^2 \equiv 1 + 5 + 20 + 60 + 120 + 120 \equiv 326 \pmod{k-5}$, which for $k = 24$ yields us with $3 \pmod{19}$. But by quadratic reciprocity, since $19 \equiv 3 \pmod 4$ and $19 \equiv 1 \pmod 3$ (so is a square modulo 3), 3 isn't a square modulo 19, so $k = 24$ is also impossible.

Finally, notice that $a_3^2 = 10 + 6a_0^2$, which is clearly a perfect square when $a_0 = 1$. We thus see that $0, 1, 3$ are the only ones that work, yielding our sum of $\frac{99 \cdot 100}{2} - 4 = 4946$.

13. Given a positive integer n with prime factorization $p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$, we define $f(n)$ to be $\sum_{i=1}^{k} p_i e_i$. In other words, $f(n)$ is the sum of the prime divisors of n, counted with multiplicities. Let M be the largest odd integer such that $f(M) = 2023$, and m the smallest odd integer so that $f(m) = 2023$. Suppose that $\frac{M}{m}$ equals $p_1^{e_1} p_2^{e_2} \cdots p_l^{e_l}$, where the e_i are all nonzero integers and the p_i are primes. Find $|\sum_{i=1}^{l} (p_i + e_i)|$.

Proposed by: Frank Lu

Answer: $\boxed{2695}$

We first find what M is. To do this, we first notice that every positive odd integer other than 1 can be written in the form $3a + 5b + 7c$, where a, b, c are non-negative integers. To do this, observe that every positive odd integer other than 1 can be written in either the form $6k+3, 6k+5$, or $6k+7$, where k is a nonnegative integer. It's not hard to see that, with these forms, we can find nonnegative integers a, b, c for each of these cases.

From here, suppose that $n \in S$ and is odd. Then, if n is divisible by any prime p that isn't one of $3, 5, 7$, then by the above, since n is odd, p is odd, so we can write $p = 3a + 5b + 7c$, where a, b, c are nonnegative integers. Replacing n with $\frac{n3^a5^b7^c}{p}$ yields another integer that lies in S. But notice that $3^a5^b7^c > p$. To see this, notice that $3^a5^b7^c \geq 3^{a+b+c} > 7(a+b+c) \geq p$, if $a+b+c \geq 3$. We can verify that $3^n > 7n$ for $n \geq 3$, by induction. As for $a+b+c = 1, 2$, observe that these are impossible, since p is odd and not one of $3, 5, 7$. But this means that M is only divisible by primes in $\{3, 5, 7\}$.

Furthermore, if n is divisible by 7^2, we can replace 7^2 with $3^3 \cdot 5$, and if n is divisible by 5^3, we can replace it with 3^5. Finally, if n is divisible by $5 \cdot 7$, then we can replace this product with 3^4. This means that M is only divisible by one of $5, 7$, and that this power needs to be either 1 or 2.

With a sum of primes being 2023, we see that our candidates are $M = 3^{672} \cdot 7$ and $M = 3^{671} \cdot 5^2$. But $3 \cdot 7 < 5^2$, meaning that $M = 3^{671} \cdot 5^2$.

We now need to find m. Notice that m must be divisible by at least 3 primes, since m is odd. We claim that $m = 3^2 \cdot 2017$. We see that this lies in S and is odd. Suppose that we have an integer n whose largest prime is p. Then, observe that $n \geq p(2023 - p)$, since a product of integers that are at least 2 is larger than their sum. However, notice that the largest primes less than 2023 that is at most 9 away from 2014 is 2017; any other choice of largest prime yields $p(2023 - p)$ is larger than 20000, whereas $9 \cdot 2017$ is smaller than this.

Therefore, $m = 3^2 \cdot 2017$, and so $M/m = 3^{669}5^2/2017$, meaning that our answer is equal to $3 + 669 + 5 + 2 + 2017 - 1 = 2695$.

14. Heron is going to watch a show with n episodes which are released one each day. Heron wants to watch the first and last episodes on the days they first air, and he doesn't want to have two days in a row that he watches no episodes. He can watch as many episodes as he wants in a day. Denote by $f(n)$ the number of ways Heron can choose how many episodes he watches each day satisfying these constraints. Let N be the 2021st smallest value of n where $f(n) \equiv 2 \mod 3$. Find N.

Proposed by: Daniel Carter

Answer: $\boxed{265386}$

Let $a(x, y)$ be the number of ways Heron can watch episodes through the xth day such that he watches at least one episode on the xth day and there are y episodes he has left to watch after the xth day. We have $a(1, 0) = 1$ and $f(n) = a(n, 0)$.

We also have the recurrence $a(x, y) = \sum_{z \geq x} a(x-1, z) + \sum_{z \geq x-1} a(x-2, z)$: the first sum counts the number of ways Heron could have watched an appropriate number of episodes assuming he watched at least one on day $x - 1$ and the second sum counts the number assuming he did not watch any episode on day $x - 1$. Given this, one can easily prove the simpler recurrence $a(x, y) = a(x-2, y-1) + a(x-1, y) + a(x, y+1)$.

Now one may iterate this recurrence many times to find $a(x, y) = a(x-2, y-1) + a(x-1, y) + \sum_{k=1}^{x-2} a(k, 0)a(x-k-1, y)$ (this process is similar to the process one may use to prove certain identities of the binomial coefficient). In particular this holds for $y = 0$, whence $f(n) = f(n-1) + \sum_{k=1}^{n-2} f(k)f(n-k-1)$.

Then one may prove the sequence $(f(n) \mod 3))_n$ is as follows: - Begin with $1, 1, 2, 1$. - Append $3 \cdot 1$ copies of 0. - Append $1, 2, 1, 1, 2, 1$. - Append $3 \cdot 4$ copies of 0. - Append $1, 2, 1, 1, 2, 1$. - Append $3 \cdot 1$ copies of 0. - etc. The number of zeros added every other step is $3 \cdot (3^{b_n} - 1)/2$, where $b_n = (1, 2, 1, 3, 1, 2, 1, 4...)$ is often known as the "ruler sequence." Then one may prove the value of n corresponding to the mth occurrence of 2 is equal to $3M$ where the ternary digits of M are the same as the binary digits of m.

In our case, $m = 2021 = 11111100101_2$, so $M = 11111100101_3 = 88462$, and our answer is $3M = 265386$.

15. Let $\triangle ABC$ be an acute triangle with angles $\angle BAC = 70^\circ$, $\angle ABC = 60^\circ$, let D, E be the feet of perpendiculars from B, C to AC, AB, respectively, and let H be the orthocenter of ABC. Let F be a point on the shorter arc AB of circumcircle of ABC satisfying $\angle FAB = 10^\circ$ and let G be the foot of perpendicular from H to AF. If $I = BF \cap EG$ and $J = CF \cap DG$, compute the angle $\angle GIJ$.

    Proposed by: Aleksa Milojevic

    Answer: $\boxed{60}$

    This problem is an instance of a more general statement which states the following (using somewhat different notation). Let A, B be the two intersections of circles Γ_1, Γ_2 and let C, D, E be points on Γ_1, F, G, H on Γ_2 such that triplets (A, C, F), (A, D, G) and (A, E, H) are collinear. Moreover, let $I = FG \cap CD, J = FH \cap CE$. Show that $\angle CJI = \angle AHG$.

    One can relate this version to the one in the problem statement by setting A, G, H as the starting triangle ABC, making Γ_2 its circumcircle and Γ_1 pass through the orthocenter.

    In the solution, we use Miquel's theorem extensively. This theorem states that for a quadrilateral $XYZT$ with $U = XT \cap YZ$ and $V = XY \cap ZT$, the circumcircles of the triangles UXY, UTZ, VYZ, VXT pass through a single point.

    Let us now begin the solution: the goal is to show that $\triangle GIJ$ is similar to $\triangle ACB$. Once we show this, it will be clear that $\angle GIJ = \angle ABC = 60^\circ$.

    As a preliminary step, note that $ADHEG$ is cyclic, as $\angle AGH = \angle AEH = \angle ADH = 90^\circ$. Let M be the intersection of this circle and of the circumcircle of ABC.

    Lemma. $FGIJM$ is cyclic.

    .....solution to be completed.....

    To show this, it is key that $BCIJ$ is cyclic. After that, it is simple to show using spiral similarity that AGH is similar to CIJ with the center in B.

    To show $BCIJ$ is cyclic, first note that $BEHJ$ is cyclic by Miquel theorem on the complete quadrilateral $ACEHFJ$. Let K be the intersection of DE and GH. K lies on IJ by Desargues. The quadrilateral $EHBK$ is cyclic because of Miquel on $ADEGHK$ and thus the pentagon $BEHJK$ is cyclic. Applying Miquel on $CDEIJK$ one gets $CIJB$ is cyclic, which completes the proof.