Individual Finals A

1. Prove that, for an arbitrary positive integer $n \in \mathbb{Z}_{>0}$, the number $n^2 - n + 1$ does not have any prime factors of the form $6k + 5$, for $k \in \mathbb{Z}_{>0}$.

2. Let $ABCD$ be a cyclic quadrilateral with circumcircle Γ, and let E be the midpoint of the diagonal BD. Let I_1, I_2, I_3, I_4 be the centers of the circles inscribed into triangles $\triangle ABE, \triangle ADE, \triangle BCE, \triangle CDE$, in that order. Prove that the circles AI_1I_2 and CI_3I_4 intersect Γ at diametrically opposite points.

   Remark: For a circle C and points $X, Y \in C$, we say that X and Y are diametrically opposite if XY is a diameter of C.

3. Alice and Bob are playing a game, starting with a binary string b of length 2022. In each step, the rightmost digit of the string is deleted. If the deleted digit was 1, Alice gets to choose which digit she wants to append on the left. Otherwise, Bob gets to choose the digit to append on the left of the string.

   Alice would like to turn the string b into the all-zero string $\underbrace{00\ldots0}_{2022}$, in the least number of steps possible, while Bob would like to maximize the number of steps necessary, or prevent Alice from doing this at all.
   a) Is there a string b for which Bob can prevent Alice in her goal, if both players play optimally?
   b) If the answer to part a is yes, find all such strings b. If the answer is no, find the maximal game time and find the set of strings b for which the game time is maximal.