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Combinatorics A Solutions

1. Select two distinct diagonals at random from a regular octagon. What is the probability that the two diagonals intersect at a point strictly within the octagon? Express your answer as $a+b$, where the probability is $\frac{a}{b}$ and a and b are relatively prime positive integers.

   Proposed by: Rishi Dange

   Answer: $\boxed{26}$

   The answer is $\frac{7}{19}$. We use complementary counting.

   Split into three cases in which the diagonals don't intersect strictly within the octagon.

   Case 1: The diagonals intersect on the octagon. For each vertex, there are 5 possible diagonals from which we must choose 2 in this case. This gives us $8\binom{5}{2} = 80$ total ways.

   Case 2: The diagonals never intersect. Take a diagonal separated by 1 vertex. There are 2 other diagonals parallel to this one. There are 4 distinct possible rotations, giving a total of $4\binom{3}{2} = 12$ possibilities. The only other way for the diagonals to not intersect is to take a diagonal separated by 2 vertices, in which case there is only 1 parallel choice and 4 rotations. Thus, we have a total of $12 + 4 = 16$ ways for this case.

   Case 3: The diagonals intersect outside of the octagon. There are two subcases here. First, we can have two non-parallel diagonals that are each separated by 1 vertex. By rotation, there are 8 ways for this. Second, we can have one diagonal separated by 1 vertex and one separated by 2. For each possible diagonal separated by 1 vertex, there are 2 possibilities for the other diagonal. Thus, we have $8 \times 2 = 16$ possibilities here. That gives us a total of $8 + 16 = 24$ ways for this case.

   The total number of ways to select diagonals that do not intersect strictly within the octagon is $80 + 16 + 24 = 120$. There are $\frac{8\times 5}{2} = 20$ diagonals, giving $\binom{20}{2} = 190$ total choices. Thus, by complementary counting, we arrive at a final answer of $1 - \frac{120}{190} = \frac{7}{19}$.

2. Eighteen people are standing in a (socially-distanced) line to enter a grocery store. Five people are wearing a black mask, 6 are wearing a gray mask, and 7 are wearing a white mask. Suppose that these 18 people got on line in a random order. The expected number of pairs of adjacent people wearing different-colored masks can be given by $\frac{a}{b}$, where $\gcd(a, b) = 1$. Compute $a+b$.

   Proposed by: Nancy Xu

   Answer: $\boxed{116}$

   For each pair of adjacent people, the probability that they are wearing different-colored masks is $\frac{5}{18}\cdot\frac{13}{17} + \frac{6}{18}\cdot\frac{12}{17} + \frac{7}{18}\cdot\frac{11}{17} = \frac{107}{153}$. There are 17 pairs of adjacent people, so the expected value is $\frac{107}{153}\cdot 17 = \frac{107}{9}$, and our answer is $107 + 9 = 116$.

3. Nelson is having his friend drop his unique bouncy ball from a 12 foot building, and Nelson will only catch the ball at the peak of its trajectory between bounces. On any given bounce, there is an 80% chance that the next peak occurs at $\frac{1}{3}$ the height of the previous peak and a 20% chance that the next peak occurs at 3 times the height of the previous peak (where the first peak is at 12 feet). If Nelson can only reach 4 feet into the air and will catch the ball as soon as possible, the probability that Nelson catches the ball after exactly 13 bounces is $2^a \times 3^b \times 5^c \times 7^d \times 11^e$ for integers a, b, c, d, and e. Find $|a| + |b| + |c| + |d| + |e|$.

   Proposed by: Rishi Dange

   Answer: $\boxed{31}$

Call an 80% bounce A and a 20% bounce B. It's easy to see that for 13 bounces to occur before ending, there must be 7 As and 6 Bs. Now we simply need to find the number of orderings of the As and Bs that are possible.

The first bounce must be B and the last two must both be A, otherwise Nelson will catch the ball before 13 bounces. For the remaining 10 slots, there must be 5 As and 5 Bs such that within those 10, there is never an instance where the number of As exceeds the number of Bs by 2 or more (not including the bounce before those 10). That way, Nelson won't catch the ball before the 13 bounce total. This is a simple Catalan counting problem that gives 132 orderings, or $C(6)$.

Thus, the final probability is $(\frac{4}{5})^7 \times (\frac{1}{5})^6 \times 132$, or $2^{16} \times 3^1 \times 5^{-13} \times 7^0 \times 11^1$.

4. There are n lilypads in a row labeled $1, 2, ..., n$ from left to right. Fareniss the Frog picks a lilypad at random to start on, and every second she jumps to an adjacent lilypad; if there are two such lilypads, she is twice as likely to jump to the right as to the left. After some finite number of seconds, there exists two lilypads A and B such that Fareniss is more than 1000 times as likely to be on A as she is to be on B. What is the minimal number of lilypads n such that this situation must occur?

   Proposed by: Austen Mazenko

   Answer: $\boxed{12}$

   This situation is modeled by a Markov chain; calculating the equilibrium distribution for each state gives the probabilities as $(1/3)^{n-2}, (1/3)^{n-3}, 2*(1/3)^{n-3}, ...2^{n-3}*(1/3)^{n-3}, (2/3)^{n-2}$. The maximum is $(2/3)^{n-2}$ and the minimum is $(1/3)^{n-3}$, and their ratio is $2^{n-2} \times 3$. This must exceed 1000, so n must be at least 12.

5. A Princeton slot machine has 100 pictures, each equally likely to occur. One is a picture of a tiger. Alice and Bob independently use the slot machine, and each repeatedly makes independent plays. Alice keeps playing until she sees a tiger, at which point she stops. Similarly, Bob keeps playing until he sees a tiger. Given that Bob played twice longer than Alice, let the expected number of plays for Alice be $\frac{a}{b}$ with a, b relatively prime positive integers. Find the remainder when $a + b$ is divided by 1000.

   Proposed by: Sunay Joshi

   Answer: $\boxed{701}$

   For simplicity, let $M = 100$. Let the random variables N_A and N_B denote the number of plays for Alice and Bob, respectively. We want to find the conditional expectation $E = E(N_A|N_B = 2N_A)$. By the Tower Law, we have

$$E = E(N_A|N_B = 2N_A) = \sum_{k=1}^{\infty} E(N_A|N_A = k, N_B = 2N_A)P(N_A = k|N_B = 2N_A) = \sum_{k=1}^{\infty} kP(N_A = k|N_B = 2N_A).$$

By definition,

$$P(N_A = k|N_B = 2N_A) = \frac{P(N_A = k, N_B = 2k)}{P(N_B = 2N_A)} = \frac{P(N_A = k, N_B = 2k)}{\sum_{\ell \geq 1} P(N_A = \ell, N_B = 2\ell)}.$$

As $P(N_A = k) = \left(\frac{M-1}{M}\right)^{k-1} \frac{1}{M}$ and $P(N_B = 2k) = \left(\frac{M-1}{M}\right)^{2k-1} \frac{1}{M}$, we find $P(N_A = k, N_B = 2k) = \left(\frac{M-1}{M}\right)^{3k} \frac{1}{(M-1)^2}$. Thus our denominator becomes

$$\sum_{\ell \geq 1} P(N_A = \ell, N_B = 2\ell) = \sum_{\ell \geq 1} \left(\frac{M-1}{M}\right)^{3\ell} \frac{1}{(M-1)^2}$$

$$= \frac{1}{(M-1)^2} \cdot \frac{\left(\frac{M-1}{M}\right)^3}{1 - \left(\frac{M-1}{M}\right)^3} = \frac{(M-1)^3}{M^3 - (M-1)^3} \frac{1}{(M-1)^2}.$$

Thus our conditional probability is given as

$$P(N_A = k | N_B = 2N_A) = \frac{\left(\frac{M-1}{M}\right)^{3k} \frac{1}{(M-1)^2}}{\frac{(M-1)^3}{M^3-(M-1)^3} \frac{1}{(M-1)^2}}$$

$$= \frac{M^3 - (M-1)^3}{(M-1)^3} \left(\frac{M-1}{M}\right)^{3k}.$$

Therefore our expected value reduces to

$$E = \sum_{k \geq 1} k \cdot \frac{M^3 - (M-1)^3}{(M-1)^3} \left(\frac{M-1}{M}\right)^{3k} = \frac{M^3 - (M-1)^3}{(M-1)^3} \sum_{k=1}^{\infty} kr^k,$$

where $r = \left(\frac{M-1}{M}\right)^3$. By differentiating the geometric series $\frac{1}{1-r} = 1+r+r^2+\ldots$ and multiplying by r, we see that $\sum_{k=1}^{\infty} kr^k = \frac{r}{(1-r)^2}$, hence the desired expectation is

$$E = \frac{M^3 - (M-1)^3}{(M-1)^3} \frac{r}{(1-r)^2} = \frac{M^3}{M^3 - (M-1)^3}.$$

For $M = 100$, the reduced fraction is $\frac{a}{b} = \frac{1000000}{29701}$. Thus $a + b \equiv 701 \pmod{1000}$, our answer.

6. Alice, Bob, and Carol are playing a game. Each turn, one of them says one of the 3 players' names, chosen from {Alice, Bob, Carol} uniformly at random. Alice goes first, Bob goes second, Carol goes third, and they repeat in that order. Let E be the expected number of names that are have been said when, for the first time, all 3 names have been said twice. If $E = \frac{m}{n}$ for relatively prime positive integers m and n, find $m + n$. (Include the last name to be said twice in your count.)

   Proposed by: Sunay Joshi

   Answer: $\boxed{383}$

   Let $E_{m,n}$ denote the expected number of *additional* names that must be said until all 3 names have been said twice, starting with m names said once, n names said at least twice, and $3 - m - n$ names said 0 times.

   We know that $E_{0,3} = 0$, and we wish to find $E = E_{0,0}$.

   We derive a recurrence relation for $E_{m,n}$. If a name is said for the first time on the next turn, then it contributes $\frac{3-m-n}{3}(E_{m+1,n}+1)$ to the expected value. Similarly, if one of the m names already said once is said again, we get a term of $\frac{m}{3}(E_{m-1,n+1} + 1)$. Finally, if one of the n names already said twice is said again, we get $\frac{n}{3}(E_{m,n} + 1)$. Adding these up, we find the recurrence

$$E_{m,n} = 1 + \frac{3-m-n}{3} E_{m+1,n} + \frac{m}{3} E_{m-1,n+1} + \frac{n}{3} E_{m,n},$$

   which simplifies to

$$\frac{3-n}{3} E_{m,n} = \frac{3-m-n}{3} E_{m+1,n} + \frac{m}{3} E_{m-1,n+1} + 1.$$

   Solving the resulting system of 9 equations, we find that $E = E_{0,0} = \frac{347}{36}$, and our answer is $m + n = 347 + 36 = 383$.

7. Cassidy has string of n bits, where n is a positive integer, which initially are all 0s or 1s. Every second, Cassidy may choose to do one of two things:

   1. Change the first bit (so the first bit changes from a 0 to a 1, or vice versa)

   2. Change the first bit after the first 1.

   Let M be the minimum number of such moves it takes to get from $1\dots1$ to $0\dots0$ (both of length 12), and N the number of starting sequences with 12 bits that Cassidy can turn into all 0s. Find $M+N$.

   Proposed by: Frank Lu

   Answer: $\boxed{6826}$

   To show that one such sequence exists, we employ induction. Our base cases are $n=1$ and $n=2$, which are clear. Now, suppose that it is possible to do this for all $n \le k$. For the string with $k+1$ bits, we can perform the following procedure using our inductive hypothesis. First, turn the first $k-1$ bits into all zeroes. Then, convert the $k+1$st bit into a zero using the second step. From here, turn the first $k-1$ bits back into ones; this is possible from the fact that both of these steps are involutions, so they their own inverses (so reversing the moves used to go from all 1s to all 0s will perform this). The result is k 1s followed by a 0, which again can be turned into all 0s by the inductive hypothesis. Thus, we have $N = 4096$.

   To compute M, notice that in order to turn our sequence into all zeroes, we need to reach the state $00\dots011$, in order to turn the last bit into a 0. Let a_n be the minimal number of moves it takes to go from $11\dots1$ to $00\dots0$, and b_n the minimal number of moves it takes to go from $00\dots01$ to $00\dots0$. Then, we see that $a_n = a_{n-2} + 1 + b_{n-1}$, using the above logic. We just need to find b_n now.

   For b_n, in order to reach all zeroes, the same logic tells us that we need to first go from $00\dots01$ to $00\dots011$, which takes b_{n-1} moves. Applying the logic above yields us that $b_n = 2b_{n-1}+1$.

   Substituting in the first recurrence yields us with the linear recurrence $a_{n+1} - 2a_n - a_{n-1} + 2a_{n-2} = 0$. Using the theory of linear recurrences, we know that this takes the form $a_n = c_1 r_1^n + c_2 r_2^n + c_3 r_3^n$, where the r_i are roots of the equation $r^3 - 2r^2 - r + 2 = 0$. But these roots are $2, 1, -1$. Therefore, we have that $a_n = c_1 2^n + c_2 + c_3(-1)^n$.

   We just need to find the value of these c_i. It's not hard to manually compute that $a_1 = 1, a_2 = 2, a_3 = 5$. Substituting these values into the above recurrence and solving the resulting system yields $a_n = \frac{2^{n+1}}{3} - \frac{1}{2} + \frac{(-1)^{n+1}}{6}$. Substituting $n = 12$ yields $a_{12} = \frac{8192}{3} - \frac{1}{2} - \frac{1}{6} = \frac{8190}{3} = 2730$. So our answer is $4096 + 2730 = 6826$.

8. Physicists at Princeton are trying to analyze atom entanglement using the following experiment. Originally there is one atom in the space and it starts splitting according to the following procedure. If after n minutes there are atoms $a_1, \dots, a_N$, in the following minute every atom a_i splits into four new atoms, $a_i^{(1)}, a_i^{(2)}, a_i^{(3)}, a_i^{(4)}$. Atoms $a_i^{(j)}$ and $a_k^{(j)}$ are entangled if and only if the atoms a_i and a_k were entangled after n minutes. Moreover, atoms $a_i^{(j)}$ and $a_k^{(j+1)}$ are entangled for all $1 \le i, k \le N$ and $j = 1, 2, 3$. Therefore, after one minute there is 4 atoms, after two minutes there are 16 atoms and so on.

   Physicists are now interested in the number of unordered quadruplets of atoms $\{b_1, b_2, b_3, b_4\}$ among which there is an odd number of entanglements. What is the number of such quadruplets after 3 minutes?

   Remark. Note that atom entanglement is not transitive. In other words, if atoms a_i, a_j are entangled and if a_j, a_k are entangled, this does not necessarily mean that a_i and a_k are entangled.

   Proposed by: Aleksa Milojevic

Answer: $\boxed{354476}$

Let G_n be the graph arising after n minutes and let $f(n)$ be the number of quadruples b_1, b_2, b_3, b_4 with an odd number of edges in their induced graph. We establish a recursive relation on $f(n)$.

The idea is to split naturally G_n into 4 parts: let $A = \{a_i^{(1)} | 1 \le i \le 4^{n-1}\}$, $B = \{a_i^{(2)} | 1 \le i \le 4^{n-1}\}$, $C = \{a_i^{(3)} | 1 \le i \le 4^{n-1}\}$, and $D = \{a_i^{(4)} | 1 \le i \le 4^{n-1}\}$.

Let b_1, b_2, b_3, b_4 be an arbitrary quadruple. We analyze this quadruple based on the sets A, B, C, D in which b_1, b_2, b_3, b_4 lie and split into cases. For example, if all $b_i \in A$, we denote this case as (A, A, A, A).

Case 1: (A, A, A, A). In this case, the answer is $f(n-1)$, inductively. Similarly, for all other cases (B, B, B, B), (C, C, C, C), (D, D, D, D) we have the same answer.

Case 2: (A, B, B, B). As the induced graphs on A, B, C, D are isomorphic, fix an isomorphism φ taking B to C. Then, the idea is that exactly half of all quadruplets on the sets (A, B, B, B) or the sets (A, C, C, C) have an odd number of edges. Why is this? If a quadruple $(b_1, b_2, b_3, b_4) \in (A, B, B, B)$ has an odd number of edges, then $(b_1, \varphi(b_2), \varphi(b_3), \varphi(b_4)$ has an even number of edges and vice versa. (This is because the existence of exactly three edges changed between the two quadruples, namely the edges between b_1 and the rest).

One can apply the same argument to all other cases of the form (A, D, D, D), (B, C, C, C), etc, to get that exactly one half of all quadruples have an odd number of edges.

Case 3: (A, A, B, C). The induced graph on A has exactly half of the edges of the complete graph on the same number of vertices. Therefore, by going through all pairs (a_1, a_2) for the A-vertices, we get again that exactly half of all possible quadruples have an odd number of edges.

Case 4: (A, A, B, B) similar to the previous one, by the same argument.

Case 5: (A, B, C, D) - all quadruples of this type work, so we have $4^{4(n-1)}$ contribution.

In total, we have $4f(n)$ for the case 1, density $1/2$ for cases 2 through 4 and $4^{4(n-1)}$ for case 5.

Therefore the final recurrence is

$$f(n) = 4f(n-1) + 4^{4(n-1)} + \frac{1}{2}\Big[\binom{4^n}{4} - 4^{4(n-1)} - 4\binom{4^{n-1}}{4}\Big],$$

with the initial terms $f(0) = 0, f(1) = 1$.