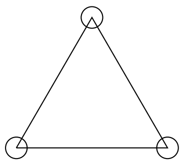
PUMaC 2020/2021 Power Round

Spring 2021



Rules and Reminders

1. Your solutions may be turned in in one of two ways:
   - You may email them to us at **pumac2019@gmail.com** by 8AM Eastern Standard Time on the morning of PUMaC, Spring, 2020 with the subject line "PUMaC 2020/2021 Power Round."
   - You may hand them in to us when your team checks in on the morning of PUMaC. Please staple your solutions together, including the cover sheet.

   The cover sheet (the last page of this document) should be the first page of your submission. Each page should have on it the **team number** (not team name) and **problem number**. This number can be found by logging in to the coach portal and selecting the corresponding team. Solutions to problems may span multiple pages, but include them in continuing order of proof.

2. You are encouraged, but not required, to use LaTeX to write your solutions. If you submit your power round electronically, **you may not submit multiple times**. The first version of the power round solutions that we receive from your team will be graded. If submitting electronically, **you must submit a PDF**. No other file type will be graded.

3. Do not include identifying information aside from your team number in your solutions.

4. Please collate the solutions in order in your solution packet. Each problem should start on a new page, and solutions should be written on one side of the paper only (there is a point deduction for not following this formatting).

5. On any problem, you may use without proof any result that is stated earlier in the test, as well as any problem from earlier in the test, even if it is a problem that your team has not solved. These are the **only** results you may use. You may not cite parts

of your proof of other problems: if you wish to use a lemma in multiple problems, please reproduce it in each one.

6. When a problem asks you to "find with proof," "show," "prove," "demonstrate," or "ascertain" a result, a formal proof is expected, in which you justify each step you take, either by using a method from earlier or by *proving* that *everything* you do is correct. When a problem instead uses the word "explain," an informal explanation suffices. When a problem asks you to "find" or "list" something, no justification is required.

7. All problems are numbered as "Problem x.y.z" where x is the section number and y is the subsection. Each problem's point distribution can be found in the cover sheet.

8. **You may NOT use any references, such as books or electronic resources, unless otherwise specified. You may NOT use computer programs, calculators, or any other computational aids.**

9. Teams whose members use English as a foreign language may use dictionaries for reference.

10. **Communication with humans outside your team of 8 students about the content of these problems is prohibited.**

11. There are two places where you may ask questions about the test. The first is Piazza. Please ask your coach for instructions to access our Piazza forum. On Piazza, you may ask any question **so long as it does not give away any part of your solution to any problem**. If you ask a question on Piazza, all other teams will be able to see it. If such a question reveals all or part of your solution to a power round question, your team's power round score will be penalized severely. For any questions you have that might reveal part of your solution, or if you are not sure if your question is appropriate for Piazza, please email us at pumac@math.princeton.edu. We will email coaches with important clarifications that are posted on Piazza.

Introduction and Advice

This year's power round is about **extremal combinatorics**, and more specifically extremal combinatorics in Graph Theory. extremal combinatorics studies the maximum and/or minimum possible cardinalities of combinatorial structures with some desired properties. It has applications in theoretical computer science, information theory, number theory, geometry and others. For example, a standard question in extremal combinatorics is of the form

If we look at structures with specific properties, how big or small can they be?

We will answer this question for a few examples with graphs.

The power round is structured such that it will walk you through proofs of some of the most important theorems. Afterwards there will be some auxiliary problems.

Here is some further advice with regard to the Power Round:

- **Read the text of every problem!** Many important ideas are included in problems and may be referenced later on. In addition, some of the theorems you are asked to prove are useful or even necessary for later problems.
- **Make sure you understand the definitions**. A lot of the definitions are not easy to grasp; don't worry if it takes you a while to fully understand them. If you don't, then you will not be able to do the problems. Feel free to ask clarifying questions about the definitions on Piazza (or email us).
- **Don't make stuff up**: on problems that ask for proofs, you will receive more points if you demonstrate legitimate and correct intuition than if you fabricate something that *looks* rigorous just for the sake of having "rigor."
- **Check Piazza often!** Clarifications will be posted there, and if you have a question it is possible that it has already been asked and answered in a Piazza thread (and if not, you can ask it, assuming it does not reveal any part of your solution to a question). **If in doubt about whether a question is appropriate for Piazza, please email us at pumac@math.princeton.edu.**
- **Don't cheat**: as stated in Rules and Reminders, you may **NOT** use any references such as books or electronic resources. If you do cheat, you will be disqualified and banned from PUMaC, your school may be disqualified, and relevant external institutions may be notified of any misconduct.

Good luck, and have fun!

– Marko Medvedev

I would like to acknowledge and thank many individuals and organizations for their support; without their help, this Power Round (and the entire competition) could not exist. Please refer to the solutions of the power round for full acknowledgments and references.

Contents

Notation

- $\forall$: for all. *ex.:* $\forall x \in \{1, 2, 3\}$ *means "for all* x *in the set* $\{1, 2, 3\}$*"*
- $A \subset B$: proper subset. *ex.:* $\{1, 2\} \subset \{1, 2, 3\}$*, but* $\{1, 2\} \not\subset \{1, 2\}$
- $A \subseteq B$: subset, possibly improper. *ex.:* $\{1\}, \{1, 2\} \subseteq \{1, 2\}$
- $f : x \mapsto y$: f maps x to y. *ex.: if* $f(n) = n - 3$ *then* $f : 20 \mapsto 17$ *and* $f : n \mapsto n - 3$ *are both true.*
- $\{x \in S : C(x)\}$: the set of all x in the set S satisfying the condition $C(x)$. *ex.:* $\{n \in \mathbb{N} : \sqrt{n} \in \mathbb{N}\}$ *is the set of perfect squares.*
- $\mathbb{N}$: the natural numbers, $\{1, 2, 3, \ldots\}$.
- $[n] = \{1, 2, 3, ..., n\}$.
- $\mathbb{Z}$: the integers.
- $\mathbb{R}$: the real numbers.
- $|S|$: the cardinality of set S.

1 Playing Billiard

Alex loves playing billiard. Recently, he learned that the billiard balls bounce off the walls of the billiard table at the same angles they come in. Further, when a ball hits a corner of the table it may chose to bounce off any line between the two lines incident to that corner. Alex is a good geometer and has precisely defined this billiard game on an arbitrary convex polygon.

Definition 1.A. A broken line $A_1A_2\dots A_n$ is a union of line segments $A_1A_2, A_2A_3, \dots A_{n-1}A_n$. We sat a broken line is closed if $A_1 = A_n$ and we call $A_1, \dots, A_n$ the breakpoints of this broken line.

Definition 1.B. Let P be a convex polygon. A broken line $A_1A_2...A_n$ is called a *billiard trajectory* on P if:

- The points A_i lie on the boundary of P, for $i = 1, ..., n$.
- If A_i lies in the interior of the edge XY of P, we have $\angle A_{i-1}A_iX = \angle YA_iA_{i+1}$.
- If A_i is a vertex of P, adjacent to the vertices X and Y of P, then

$$|\angle XA_iA_{i-1} - \angle A_{i+1}A_iY| + \angle XA_iY \leq 180^\circ$$

Now, Alex is interested what kinds of trajectories he can find on a polygon, and has one more definition.

Definition 1.C. Let $A_1A_2...A_{n+1}$ be a billiard trajectory on a polygon P. If $A_1 = A_n$ and $A_2 = A_{n+1}$, then $A_1...A_n$ is called a *closed trajectory*. For a closed trajectory C consisting of n line segments, we define its order as $|C| = n$.

Clearly, if there is at least one closed trajectory C on a polygon P, there is infinitely many of them, obtained by walking several times over C. Trajectories obtained by repeating several times the same basic closed trajectory C are called *powers* of C. Trajectories that cannot be obtained as powers are called *prime*.

Theorem 1.I. For any convex polygon P there exist infinitely many prime closed billiard trajectories on P.

We will prove this theorem in several steps. The main idea is to take a longest closed broken line L with vertices on the boundary of P and at most n vertices, for some wisely chosen n.[1] The following three problems will show that this line satisfies the conditions of the theorem 1.I.

Problem 1.1. Prove that L has exactly n vertices.

[1] Although it may not be utterly obvious that such a maximal broken line exists, this follows by a simple compactness argument. However, as this is not our topic here, you may assume without proof that such a maximal broken line exists.

Proof 1.1. Assume, for the sake of contradiction, that L has less than n vertices, and let its vertices be $A_1, A_2, \ldots, A_m, A_{m+1} = A_1$. Then, we can construct a longer closed broken line L' by considering an arbitrary point X on the boundary of P different from A_1, A_2 and setting the vertices of L' to be $A_1, X, A_2, \ldots, A_m, A_{m+1} = A_1$. The broken line L' still has at most n vertices, because $m < n$ and it is longer than L because $A_1X + XA_2 > A_1A_2$ by the triangle inequality. Therefore, we conclude that L is not of maximal possible length and we obtain a contradiction. Therefore, we conclude L must have n vertices.

Problem 1.2. Prove that L is indeed a closed billiard trajectory, in the sense of definitions 1.B and 1.C.

Proof 1.2. In order to show L is a closed billiard trajectory, it will be enough to check that the latter two conditions of the definition 1.B are satisfied, as all other conditions directly follow from the definition of L. This proof will have two parts: first, we will show that all breakpoints of L are vertices of P, and then we will prove the property three of the definition 1.B for these points. These two parts are very similar in nature, but we present two different arguments. Note that both of these arguments can essentially be used to prove both parts. We use both approaches as to show several possible options students could take.

First, we will show that all the breakpoints of L are actually the vertices of P. To see this, we will take an arbitrary breakpoint A_i and look at the segments $A_{i-1}A_i$ and A_iA_{i+1}. As the length of L is maximal, we see that A_i is a point of P which maximizes $A_{i-1}A_i + A_iA_{i+1}$, for given A_{i-1}, A_{i+1}. We claim that it is impossible that such A_i is in the interior of some edge of P. If A_i was in the interior of the edge XY, we would consider the set of all the points T with $A_{i-1}T + TA_{i+1} = A_{i-1}A_i + A_iA_{i+1}$, which is an ellipse containing A_i. As XY is a segment through A_i and as the ellipse is convex, at least one of X, Y must lie outside of the ellipse. Say this point is X. Then, $A_{i-1}X + XA_{i+1} > A_{i-1}A_i + A_iA_{i+1}$, contradicting the fact $A_{i-1}A_i + A_iA_{i+1}$ was maximal. Thus, A_i must be a vertex of the polygon.

Now, we will prove the property three of the definition 1.B. Let A_i be a breakpoint, which is a vertex of P and let X, Y be the adjacent vertices. Assume, without loss of generality, that $\angle XA_iA_{i-1} \leq XA_iA_{i+1}$. Moreover, for the sake of contradiction, assume that $|\angle XA_iA_{i-1} - \angle A_{i+1}A_iY| > 180^\circ - \angle XA_iY$. Again, because of symmetry, we may assume $\angle XA_iA_{i-1} > \angle A_{i+1}A_iY$ and so we have:

$$\angle XA_iA_{i-1} - \angle A_{i+1}A_iY > 180^\circ - \angle XA_iY$$

Let s be the external angle bisector of $\angle A_{i-1}A_iA_{i+1}$, and let B be the reflection of A_{i+1} over s. First, we show that points A_{i-1}, A_{i+1} lie on the same side of s, and that the line s separates them from X. The first part is clear by definition of s, and for the second part, we need to prove that $\angle XA_iA_{i-1} > \angle(s, A_iA_{i+1})$. Further, we compute the angles to get

$$\angle(s, A_iA_{i+1}) = 90^\circ - \frac{1}{2}\angle A_{i-1}A_iA_{i+1} = 90^\circ - \frac{1}{2}\big(\angle XA_iY - \angle XA_iA_{i-1} - \angle A_{i+1}A_iY\big)$$

Thus, the previous inequality transforms into:

$$\begin{aligned}
&\angle XA_iA_{i-1} > \angle(s, A_iA_{i+1}) \iff \\
&\angle XA_iA_{i-1} > 90^\circ - \frac{1}{2}\big(\angle XA_iY - \angle XA_iA_{i-1} - \angle A_{i+1}A_iY\big) \iff \\
&\frac{1}{2}\big(\angle XA_iA_{i-1} - \angle A_{i+1}A_iY\big) > 90^\circ - \frac{1}{2}\angle XA_iY,
\end{aligned}$$

where the last line follow by assumption we made.

Now, we will show a final contradiction by showing that $A_{i-1}X + XA_{i+1} > A_{i-1}A_i + A_iA_{i+1}$. To show this, note that $A_{i-1}A_i + A_iA_{i+1} = A_{i-1}A_i + A_iB = A_{i-1}B$, because A_{i-1}, A_i and B are collinear. Moreover, as s is the perpendicular bisector of $A_{i+1}B$, we see that $A_{i+1}X > BX$, because X and B are on the same side of s. Thus, we can claim the following chain of inequalities:

$$A_{i-1}A_i + A_iA_{i+1} = A_{i-1}B < A_{i-1}X + XB < A_{i-1}X + XA_{i+1}.$$

This shows that A_i does not maximize the sum $A_{i-1}A_i + A_iA_{i+1}$, which gives an ultimate contradiction and means $|\angle XA_iA_{i-1} - \angle A_{i+1}A_iY| \le 180^\circ - \angle XA_iY$.

Problem 1.3. Complete the proof of theorem 1.I.

Proof 1.3. In order to complete the proof, we pick a arbitrary prime number n and perform the construction as in the previous two problems. The resulting trajectory L is clearly a billiard trajectory. Moreover, it cannot be a power of another prime trajectory C, because we would have that the order of C divides n, which is not possible as n is prime. Thus, L must be prime. As there are infinitely many primes, there are also infinitely many prime trajectories.

A similar idea can be applied to prove another result of the same flavor.

Problem 1.4. Prove that, given any two points A, B on the boundary of a convex polygon P, there exists infinitely many billiard trajectories starting at A and ending at B.

Proof 1.4. Let n be an arbitrary integer. It suffices to take the longest possible broken line L which starts at A, ends at B and has at most n breakpoints. It is clear, by an argument identical to proof proof 1.1, that L has n breakpoints. Let these breakpoints be $A = A_1, A_2, \ldots, A_n = B$. Moreover, note that given A_{i-1}, A_{i+1} the point A_i is chosen so that $A_{i-1}A_i + A_iA_{i+1}$ is maximized. Therefore, by the same argument as in proof 1.2, one concludes that all of A_i (except maybe A_1 and A_n) are vertices of P and that L is a billiard trajectory, starting at A and ending at B.

2 Introduction to ant-paths

Alan the Ant lives on the surface of a three-dimensional polyhedron $\mathcal{P}$. His life consists of moving along this surface, in a specific manner. Every day, Alan chooses two points X and Y on that surface, positions himself at X and tries to get as fast as possible to Y. To do that, he needs to find a shortest path between X and Y on the surface of $\mathcal{P}$.

To formalize these concepts, we have the following definitions. Although these definition may seem cumbersome at first, they are only rephrasing the intuition we have about polyhedra and their surfaces in mathematical terms. The first definition gives us the concept of the polyhedron:

Definition 2.A. A *half-space* is a region on one side of the plane. Half-spaces can be *closed* or *open*, depending on whether they contain the bounding plane or not. A set of points in $\mathbb{R}^3$ is said to be *bounded* if there is a big enough ball containing it. A *polyhedron* is a bounded intersection of several closed half-spaces.

The following definition will formalize the concept of the surface of that polyhedron.

Definition 2.B. Let $\mathcal{P}$ be a polyhedron and let H be a plane which does not intersect the interior of $\mathcal{P}$. If the intersection of H and $\mathcal{P}$ is a point, that point is called a *vertex* of $\mathcal{P}$. If $H \cap \mathcal{P}$ is a line segment, that line segment is called an *edge* of $\mathcal{P}$, while if $H \cap \mathcal{P}$ is a plane polygon, it is called a *face* of $\mathcal{P}$. The set of vertices of $\mathcal{P}$ is denoted by $V(\mathcal{P})$. The *surface* of $\mathcal{P}$ is the union of all its faces. We denote the surface of $\mathcal{P}$ by $S(\mathcal{P})$.

Finally, the last definition explains what a path on the surface is and how we measure its length.

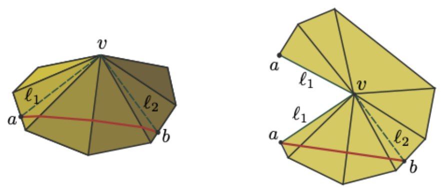
Definition 2.C. Let X and Y be two points on the surface of a polyhedron $\mathcal{P}$. A *path* between X and Y on the surface $S(\mathcal{P})$ is a continuous curve $C \subset S(\mathcal{P})$, having one end in X and the other end in Y. For any such curve C, we can pick several points along it, say $X = T_0, T_1, T_2, ..., T_k = Y$ in that order along C, such that the segments of C between T_i and T_{i+1} lie on only one face. Then, as the segments T_i and T_{i+1}, we can measure their length and denote it by $l(T_iT_{i+1})$. Now, we can define the *length* of C as $l(C) = l(T_0T_1) + l(T_1T_2) + \cdots + l(T_{k-1}T_k)$. A *shortest path* between X and Y is the path between X and Y with the minimum length.

It turns out that shortest paths on a surface of a polyhedron are exceedingly interesting, as the following properties show:

Problem 2.1. Let C be the shortest path between points X and Y on the surface of polyhedron $\mathcal{P}$. Prove that the interior of C does not contain any vertices of $\mathcal{P}$. In other words, C contains no vertices of $\mathcal{P}$ except maybe X and Y.

Hint: Argue by contradiction. Assume there is a shortest path C between X and Y going through the vertex V and use it to construct a path from X to Y shorter than C to get a contradiction.

Proof 2.1. For the sake of contradiction assume otherwise, that there is a shortest path C between X and Y that contains a vertex v of $\mathcal{P}$. Let a and b be two points on the path C which are on two faces which contain v. Note that since the polyhedron is convex, the sum of the face angles at v is less than 2π. Then consider cutting open the polyhedron at v along the line av, and unrolling it onto a plane. The convexity makes this possible. Notice that since C is the shortest path from X and Y, ab has to be the shortest path between a and b. But notice that $av + bv > ab$ by the triangle inequality. Hence if we take the line segment ab and find a path on $\mathcal{P}$ which results from ab when the "cutting" is reversed, and call that path from a to b on the surface of the polyhedron γ we have the following. Let C' be the path resulting by taking C from X to a then γ and then C from b to Y, then C' is a path from X to Y that is shorter than C, which contradicts the fact that C is the shortest path. This proves that our assumption that v is on C was wrong.



Problem 2.2. Let C be the shortest path between points X and Y on the surface of polyhedron $\mathcal{P}$. Given an edge AB of $\mathcal{P}$, show that C intersects AB only in isolated points. In other words, show that C does not contain a subsegment of AB of nonzero length. Moreover, show that whenever C intersects AB, it changes the face (i.e. C uses the edge AB only when it goes from one face to the other).

Hint: Argue by contradiction, as before. Assume that some forbidden structure exists, and use it to construct a shorter path.

Proof 2.2. For the sake of contradiction assume otherwise, that there is a shortest path C between X and Y that contains a subsegment of an edge AB it crosses of nonzero length. Let the points a, b be on AB such that ab is a part of C, and assume that it is of nonzero length. Let then α, β be points on C which are before a and after b respectively, such that they are close enough to a and b such that C doesn't change a face from α to a and from b to β. Firstly, assume that α and β are on different faces. Then "unfold" $\mathcal{P}$ along the edge AB, that is take a rotation that sends one of the faces incident to AB to the the plane of the other face. Let this rotation R take β to β' and fix α. Then let $\alpha\beta' \cap AB = \gamma$. Then notice that by the triangle inequality we have

$$\alpha A + AB + B\beta' = \alpha A + A\gamma + \gamma B + B\beta' > \alpha\gamma + \gamma\beta'$$

And hence when applying the reverse rotation R^{-1}, we get that taking the path from α to γ to β is shorter than including the segment AB, a contradiction with the minimum length of C, which contradicts our assumption.

In the second case, we assume that α, β are on the same face. Then notice that by the triangle inequality $\alpha\beta < \alpha A + AB + B\beta$, and so it's shorter to just take the segment $\alpha\beta$, which would contradict the minimal length of C.

The first part proves that if C intersects an edge it intersects it in one point. The second part proves that if it intersects an edge, the path C has to cross to a different face.

Problem 2.3. Let C be the shortest path between points X and Y on the surface of polyhedron $\mathcal{P}$. Given an edge AB of $\mathcal{P}$ we pick a subsegment KL of C which intersects AB at exactly one point, T. Show that $\angle KTA = \angle BTL$ (the angles are measured in their respective planes).

Proof 2.3. (Here we need to assume that the path stays on the same face from K to T and from T to L). Then consider "unfolding" the polyhedron along the edge AB, that is apply a rotation R to the face which contains L into the plane of the face which contains L. Let L' be the image of L under that rotation. Since C is the shortest path from X to Y, KL is the shortest path from K to L, and hence it unfolds into a straight line segment KL'. Otherwise, applying R^{-1} to the straight line segment KL' we would get a shorter path from K to L. But then $\angle BTL = \angle BTL'$, since R is an isometry. Furthermore, since the line segments KL' and AB intersect in T, we have that $\angle KTA = \angle BTL'$, which proves the desired claim.

The above nice properties motivate the following definition, which generalizes the concept of shortest paths, while maintaining all its useful properties.

Definition 2.D. Let $\mathcal{P}$ be a polyhedron and $S(\mathcal{P})$ its surface. An *ant-path* on $S(\mathcal{P})$ is a piecewise linear curve C which satisfies the following: for every point P in the interior, there is a subsegment $X_P Y_P$ of C which contains P, and on which C agrees with the shortest path between X_P and Y_P.

In other words, an ant-path is a path which looks like a shortest path when zoomed in enough. As we shall see in the next problem, the properties that we proved for the shortest paths above are enough to guarantee something is an ant path.

Problem 2.4. Let C be a piecewise linear curve on the surface $S(\mathcal{P})$ of the polyhedron, with breaking points $T_1, T_2, ..., T_k$. Prove that C is an ant-path if and only if it satisfies the following three properties:

- All of the breaking points $T_1, T_2, ..., T_k$ lie on the edges of $\mathcal{P}$,
- C contains no vertices in its interior,
- If the breaking point T_i is contained in a short subsegment KL (K and L being on the different faces of $\mathcal{P}$) of C, and on the edge AB of $\mathcal{P}$, then $\angle KT_iA = \angle BT_iL$.

Remark: Formally, to solve this problem you need to prove two directions. However, you will find one of them easy due to the work we already did.

Proof 2.4. NEED TO WRITE

Of special interest are ant-paths that do not have beginnings or ends: the closed ant-paths.

Definition 2.E. A *closed* ant-path is a ant-path C whose beginning and end coincide. If this point is denoted by P, C needs to satisfy: there is a subsegment X_PY_P of C which contains P, and on which C agrees with the shortest path between X_P and Y_P. A closed ant-path is *simple* if it has no self-intersections.

Interestingly enough, the simple closed ant-paths are relatively rare. The following problem shows that generic tetrahedrons almost never contains simple closed ant-paths.

Problem 2.5. Let $A_1A_2A_3A_4$ be a tetrahedron, and let θ_i denote the sums of angles at the vertex A_i (in other words, $\theta_1 = \angle A_2A_1A_3 + \angle A_2A_1A_4 + \angle A_4A_1A_3$, and similarly for the other indices). If $\theta_i + \theta_j \neq 2\pi$ for all $i \neq j \in [4]$, then there is no simple closed ant-path with 3 or 4 segments on $S(A_1A_2A_3A_4)$.

Hint: Try to draw such a path and show it cannot exist due to angle constraints given by the problem 2.4.

Proof 2.5. (Solution for 3 or 4 breaking points.) Suppose the contrary, that is there is γ a closed simple and path with 3 or 4 breaking points.

We first do the case that γ, the closed simple ant path, is abc, where a, b, c are points on three of the sides of the tetrahedron which share a vertex. Let v_1 be the vertex that all three of the edges contain, and let α_1 be the pointiness at v_1. Note that the sum of all face angles in a tetrahedron is 4π (4 triangles with π each.) Then consider the sum of angles in the tetrahedron v_1abc. On the hand it is 4π. However, since γ is an ant path, we have shown the angle $\angle cav_1$ is equal to the exterior angle bav_1 by problem 2.1.3, that is $\angle cav_1 + \angle bav_1 = \pi$. From this, we get that the sum of all angles is π for each of the vertices a, b, c, π for the triangle abc, and finally α_1 for the sum of the face angles at v_1. Hence we get

$$4\pi = \pi + 3\pi + \alpha_1$$

Which is a contradiction as $\alpha_1 > 0$.

Now we do the case that γ has 4 breaking points, that is it is $abcd$, where the points are on four different sides of the tetrahedron (no 3 of which are on the same face). Then take v_1, v_2 to be two vertices of the tetrahedron on one side of the path. Let α_1, α_2 be the pointiness of each of the vertices, respectively. Then let Σ be the total sum of angles in the quadrilaterals v_1abv_2, v_2cdv_1 and triangles v_1ad and v_2bc. Clearly $\Sigma = 6\pi$. On the other hand, since γ is an ant-path, the sum of the angles $\angle dav_1 + \angle bav_1 = \pi$, since the angles dav_1 and the exterior angle bav_1 are the same by problem 2.1.3. Similarly at each of the vertices a, b, c, d the sum of the angles in the mentioned triangles and quadrilaterals is π. The other angles sum up to $\alpha_1 + \alpha_2$.

$$\Sigma = 4\pi + \alpha_1 + \alpha_2$$

And hence $\alpha_1 + \alpha_2 = 2\pi$, which is a contradiction with the assumption that no two pointiness sum up to 2π.

The above result can be generalized to many other polyhedra. In some sense, almost no generic polyhedra have simple closed ant paths on their surfaces. However, in order to prove that, we will need to develop a stronger machinery.

3 More about polyhedra

We will now recall some of the well-known properties of the polyhedrons, and also introduce some on the new ones. One of the most famous results describing 3-dimensional polyhedra is the celebrated Euler's formula:

Problem 3.1. Let $\mathcal{P}$ be a polyhedron having V vertices, E edges and F faces. The following formula relates these three quantities: $V - E + F = 2$.

Hint: Form a planar graph out of this polyhedron and sum up its angles.

Proof 3.1. We will first choose a point X very close to a face of the polyhedron. We will then project the polyhedron on an arbitrary plane, by sending each point $P \in \mathcal{P}$ to the intersection of XP with the plane.

The resulting configuration is a set of vertices in the plane, connected by edges and forming faces. This structure can be considered as a graph, noticing that we get the same number of vertices, edges and faces as in the starting polyhedron. It is worth noting that one of the faces of the resulting graph will be unbounded (i.e. everything "outside" the graph will constitute one face).

The sum S of angles of the bounded faces can now be computed in two different ways. The first way to evaluate this sum is to add up the angles at each vertex. We can notice that the sum of angles around every vertex v not in the unbounded face is 2π. Their total sum is then $2\pi m$, where m is the number of "internal" vertices. The sum of all the angles of the vertices of the unbounded face is $\pi(n-2)$, where n is the number of vertices of the unbounded face. This gives $S = 2\pi m + \pi(n-2) = 2\pi V - n\pi - 2\pi$.

On the other hand, we can sum the angles in every face. Let f be a face, and let f contain $|f|$ edges. Then the sum of angles in f is $\sum_i \alpha_i = \pi(|f|-2)$. Therefore, the sum of all angles is $\sum_{f \text{ bounded}} \pi(|f|-2) = \pi \sum_{f \text{ bounded}} |f| - 2\pi(F-1)$. When summing $\sum_{f \text{ bounded}} |f|$, all the edges are counted twice, except those on the boundary of the unbounded face, which are counted once. Thus, we get $S = 2\pi E - \pi n - 2\pi(F-1)$.

This gives $2\pi E - 2\pi F + 2\pi = 2\pi V - 2\pi$, and therefore $E - F + 1 = V - 1$, which directly implies $V - E + F = 2$

As we aim to generalize the result of problem 2.5, we will follow a similar path, and thus define the following quantity:

Definition 3.A. For a vertex v of the polyhedron $\mathcal{P}$, we define its *pointiness* as $p(v) = 2\pi - \sum_i \alpha_i$, where α_i are the face angles of $\mathcal{P}$ at the vertex v.

Although the following definition may seem slightly arbitrary at first, the following claim shows that it is actually useful:

Problem 3.2. For a polyhedron $\mathcal{P}$, we have $\sum_{v \in V(\mathcal{P})} p(v) = 4\pi$.

Proof 3.2. First, we apply the definition of $p(v)$ to get $\sum_{v \in V(\mathcal{P})} p(v) = \sum_{v \in V(\mathcal{P})} 2\pi - \sum_i \alpha_{i,v} = 2\pi V - \sum_{i,v} \alpha_{i,v}$. But the latter sum is actually equal to the the sum of all face angles of the polyhedron, and can thus be interpreted as $\sum_{f:\text{ face of } P} \sum_i \alpha_{i,f} = \sum_{f:\text{ face of } P} (|f| - 2)\pi$, where $|f|$ denotes the number of sides f has. Thus, we only need to prove:

$$2\pi V - \sum_f (|f| - 2)\pi = 4\pi$$

Now, the sum $\sum_f |f|$ counts the number of pairs (f, e) where f is a face of the polygon and e is an edge of f. On the other hand, as there are precisely two faces containing each edge, the number of pairs is also $2E$. Therefore, we have

$$2\pi V - \sum_f (|f| - 2)\pi = 2\pi V - 2\pi E + \sum_f 2\pi = 2\pi(V - E + F) = 4\pi,$$

as required. Thus, we have $\sum_{v \in V(\mathcal{P})} p(v) = 4\pi$. □

The preceding problem reminds us of two dimensional case, which we can perhaps solve first in order to build intuition.

Problem 3.3. Let $\mathcal{M}$ be a convex plane polygon, with vertices $X_1, ..., X_k$. For every vertex X_i, we can define the *outside angle* at X_i as π minus the angle of $\mathcal{M}$ at X_i. Then, the sum of outside angles is 2π.

Proof 3.3. We will prove this by induction on k. In case $k = 3$, it is clear that the sum of angles of the triangle is π, and that the sum of outside angles is 2π.

To perform the inductive step, consider the polygon $\mathcal{M}$ as described in the problem statement and let $\mathcal{M}'$ be the polygon on the vertices $X_1, \ldots, X_{k-1}$. Moreover, we will denote the angle of $\mathcal{M}$ at X_i by α_i and we will set $\beta_i = \pi - \alpha_i$ to be the outside angle at X_i. Then, the inductive hypothesis on $\mathcal{M}'$ gives

$$(\beta_1 + \angle X_{k-1}X_1X_k) + \sum_{i=2}^{k-3} \beta_i + (\angle X_k X_{k-1} X_1 + \beta_{k-1}) = 2\pi.$$

It is clear that $\beta_k = \pi - \alpha_k = \angle X_{k-1}X_1X_k + \angle X_kX_{k-1}X_1$. Therefore, we have:

$$\sum_{i=1}^{k} \beta_i = \sum_{i=1}^{k-1} + \angle X_{k-1}X_1X_k + \angle X_kX_{k-1}X_1 = 2\pi,$$

which completes the proof.

After having learned how pointiness of vertices behave, we can link that behaviour with ant-paths.

Problem 3.4. Let C be a simple closed ant-path on the surface of $\mathcal{P}$. It divides the surface $S(\mathcal{P})$, and the vertices of $V(\mathcal{P})$ consequently, into two parts. The sum of pointiness of vertices in each part is equal to 2π.

Proof 3.4. We will consider only one of the two parts and show the sum of the pointiness in that part is equal to 2π. As the sum of all pointiness in 4π, the sum in the other part must be 2π as well.

Let S_1 denote one of the parts, and let $M_1, M_2, \ldots, M_k$ be the faces of that part (note that some of M_i may not be faces, but parts of faces of $\mathcal{P}$ intersected by C). Let n_i denote the number of vertices of M_i, and let $\alpha_{i,1}, \cdots, \alpha_{i,n_i}$ be the angles of M_i. Finally, let C touch l of M_is.

The idea is to compute the sum of all the angles $\alpha_{i,j}$ in two different ways. Therefore, let $S = \sum_{i=1}^{k}\sum_{j=1}^{n_i}\alpha_{i,j}$ and note that:

$$S = \sum_{i=1}^{k}\sum_{j=1}^{n_i}\alpha_{i,j} = \sum_{i=1}^{k}(n_i - 2)\pi$$

On the other hand, we may group $\alpha_{i,j}$ by the vertex of S_1 they touch. The sum of angles around points of S_1 that lie on C is exactly π, because C is an ant-path. The sums of angles around other vertices of S_1 are $2\pi - p(v)$, by the definition of the pointiness. Therefore, we get $S = l\pi + \sum_{v\in S_1} 2\pi - p(v)$. Equating the two expressions for S gives a way to compute $\sum_{v\in S_1} p(v)$ in the following way:

$$\sum_{i=1}^{k}(n_i - 2)\pi = l\pi + \sum_{v\in S_1} 2\pi - p(v)$$

$$\sum_{v\in S_1} p(v) = \pi(2V(S_1) - \sum_{i=1}^{k}(n_i - 2) + l)$$

Therefore, it suffices to show that $2V(S_1) - \sum_{i=1}^{k}(n_i - 2) + l = 2$. We will be able show this using the Euler formula, applied on the graph we obtain by projecting S_1 onto the plane. The number of vertices of this graph is $V(S_1) + l$, the number of its edges is $\frac{\sum_{i=1}^{k} n_i + l}{2}$ (here, we used the formula $\sum_f |f| = 2E$ established earlier) and the number of its faces is $k + 1$ (because we count the outside face as well). Therefore, we have:

$$V(S_1) + l - \frac{\sum_{i=1}^{k} n_i + l}{2} + k + 1 = 2$$

$$V(S_1) + \frac{l}{2} - \frac{(\sum_{i=1}^{k} n_i) - 2k}{2} = 1$$

$$2V(S_1) - \sum_{i=1}^{k}(n_i - 2) + l = 2$$

This completes the proof. □

The previous problem now gives an even simpler solution to problem 2.5. As there is no way to find two vertices with pointiness adding up to 2π (by the constraints of the problem), there are no simple closed ant-paths on the surface of the tetrahedron. Furthermore, the previous claim gives us a very strong tool when showing that a given polyhedron has no simple closed ant-paths - it is enough to check that no subset of its vertices has the sum of pointiness equal to 2π. In some sense, this means that the most of polyhedra do not have simple closed ant-paths on their surfaces (making these concepts more precise would be out of scope of this power round).

The inverse of the previous theorem does not hold, as we will show in the following problem.

Problem 3.5. Explicitly construct a polyhedron $\mathcal{P}$ with no simple closed ant-paths such that there is a set of its vertices $V_0 \subset V(\mathcal{P})$ in which $\sum_{v\in V_0} p(v) = 2\pi$.

Proof 3.5.

There is one more, somewhat surprising result, concerning different ant-paths on polyhedrons:

Problem 3.6. Let $\mathcal{P}$ be a polyhedron with two different simple closed ant-paths C_1 and C_2 which do not intersect. Then, these ant-paths have the same length.

Proof 3.6. As C_1, C_2 do not intersect, they split $S(\mathcal{P})$ in three regions, say S_1, S_2 and S_3, the region S_2 being between C_1 and C_2. As $\sum_{v \in S_1} = 2\pi$ and $\sum_{v \in S_3} = 2\pi$ by the previously established claims, we see that the region S_2 contains no vertices. In particular, that means that C_1 and C_2 intersect the same edges and the same faces in the same order. Let $F_1, F_2, \ldots, F_k$ be the faces C_1 and C_2 intersect, and assume they intersect them in that precise order. The idea is to unfold the polyhedron onto the plane containing F_1, by first unrolling the face F_2, then F_3 and so on until F_k.

In this way, the ant-paths C_1 and C_2 unroll onto straight line segments. Suppose C_1 and C_2 start at edge e between F_1 and F_2. Then, the unroll until e falls on the original plane once more. It is clear that both appearances of e are parallel to each other, and that the intersections of C_1 and C_2 with them are at the same distance. This means that C_1 and C_2 unroll onto opposites sides of a parallelogram, which further means they have to be of equal length.

4 Ant-paths on tetrahedra

In this section we will examine the simple closed ant-paths on the surfaces of various tetrahedra. We start off by examining and classifying the ant-paths on the regular tetrahedron.

We are interested in a question of the form: what are all of the possible lengths of ant-paths on the surface of the regular tetrahedron. First, we will notice that a tetrahedron has a very special property that makes it even simpler than other regular polyhedra.

Problem 4.1. Let $ABCD$ be a regular tetrahedron, assume the face ABC is horizontal, and denote this plane by α. Prove that it is possible to roll the given tetrahedron on this plane such that the faces of the tetrahedron form a triangular tiling of the plane. Moreover, prove that it is possible to assign letters a, b, c, d to the vertices of this tiling such that the vertex A of the tetrahedron always lands on the vertex of the tiling marked by a, and similarly for the vertices B, C, D.

Proof 4.1. (Note for Graders: we can just check a drawing) The images of the vertices will form the subdivision of the plane with vertices

$$V = \left\{ v_{n,m} = \left(\frac{n}{2}, \frac{\sqrt{3}}{2} m \right) : n \equiv m \pmod 2 \right\}.$$

It is an inductive argument to show that we can label the vertices as

$$\begin{aligned}
A &= \{v_{n,m} \in V : n \equiv m \equiv 0 \pmod 4 \text{ or } n \equiv m \equiv 2 \pmod 4\} \\
B &= \{v_{n,m} \in V : n \equiv 2, m \equiv 0 \pmod 4 \text{ or } n \equiv 0, m \equiv 2 \pmod 4\} \\
C &= \{v_{n,m} \in V : n \equiv m \equiv 1 \pmod 4 \text{ or } n \equiv m \equiv 3 \pmod 4\} \\
D &= \{v_{n,m} \in V : n \equiv 1, m \equiv 3 \pmod 4 \text{ or } n \equiv 3, m \equiv 1 \pmod 4\}.
\end{aligned}$$

Having constructed this useful rolling of the tetrahedron on the plane, we can start dealing with ant-paths. Assume the $S(ABCD)$ contains an ant-path $\mathcal{C}$. Roll the tetrahedron $ABCD$ along this ant-path, until we come back where we started on the ant-path. In the plane, this ant-path will have the following form: its endpoints will be on the edges of the tiling marked by the same letters, and oriented the same way (e.g. both endpoints will be on the edges ab of the tiling, and a is left of b on both edges). Moreover, as long as such a plane segment does not contain any vertices of the tiling, it will be possible to uniquely bring it back on the surface of $ABCD$.

Problem 4.2. Prove that all closed ant-paths on the surface of $ABCD$ are simple.

Hint: How are the cells of the tiling corresponding to the same face oriented?

Proof 4.2. When we map a closed geodesic back onto plane via the rolling map in Problem 4.1, the subsegments of the geodesic that are contained in the same face are parallel. Hence, if there is an intersection the path would continue in the same way as before.

Now, we are able to produce an answer to our starting question:

Problem 4.3. Assume tetrahedron $ABCD$ has unit edge length. Find all possible lengths of closed ant-paths.

Proof 4.3. Let $\omega = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$. Then $\mathbb{Z}\omega + \mathbb{Z}$ represent the set of points in $\mathbb{R}^2$ representing the vertices of the rolled tetrahedron. Moreover, each such point can be represented uniquely by a integral linear combination of $1, \omega$. Without loss of generality, let our closed geodesic start at $(\alpha, 0)$ with $\alpha \in (1, 0)$. The equivalence class of this point is

$$\{(\alpha + 2n + m, -\sqrt{3}m), (-\alpha + 2n + m, -\sqrt{3}m) : n, m \in \mathbb{Z}\}.$$

The geodesic cannot end up in a point of the second type since it goes through the point $n(1, 0) + m(1/2, \frac{\sqrt{3}}{2})$. In the first case, the distance is $2\sqrt{m^2 + mn + n^2}$. It suffices to find the $m, n \in \mathbb{Z}$ that gives a irreducible geodesic. If m, n are not coprime, we can divide by their gcd to reduce the geodesic. Now suppose $\gcd(m, n) = 1$. Then distinct (m, n) determine different geodesics. The condition for giving the same geodesic would be the slopes are equal:

$$\frac{-\sqrt{3}m}{2n + m} = \frac{-\sqrt{3}m_0}{2n_0 + m_0} \implies mn_0 = m_0 n.$$

From the coprime relations, we get $m = m_0, n = n_0$.

Now, having solved the main question in case of regular tetrahedron, we will broaden our focus. There is one specific family of tetrahedra, called *equihedral* tetrahedra which behave very nicely with respect to ant-paths. These tetrahedrons are defined as follows:

Definition 4.A. A tetrahedron $ABCD$ is called *equihedral* if its four faces ABC, ADC, ADB, and CDB are all congruent.

A useful way to visualize equihedral tetrahedra is as the face diagonals of a rectangular prism:

Problem 4.4. Prove that if $ABCD$ is an equihedral tetrahedron, there exists a rectangular prism where four of its eight vertices are A, B, C, and D, and the six edges of $ABCD$ are diagonals of the six faces of the prism.

Proof 4.4. (Sketch) After a rigid motion, we can let $A = (x, 0, 0), B = (0, y, 0), C = (0, 0, z)$ where $x, y, z \in \mathbb{R}_{\geq 0}$ are to be decided. These variables satisfy the system of equations

$$\begin{aligned} x^2 + y^2 &= AB^2 \\ y^2 + z^2 &= BC^2 \\ z^2 + x^2 &= CA^2 \end{aligned}$$

which has a unique solution. The prism spanned by A, B, C, O is the desired rectangular prism.

There are many other statements which are equivalent to the given defintion of a tetrahedron being equihedral:

Problem 4.5. Let Δ be a tetrahedron with vertices $ABCD$. Prove that the following statements are equivalent:

- Δ is equihedral in the sense of Definition 4.A.
- The perimeters of all faces ABC, ABD, ACD, and BCD are equal.
- The pointiness of all vertices are equal, i.e. $p(A) = p(B) = p(C) = p(D)$.
- The dihedral angles at the opposite edges are equal. In other words, the angle between planes ABC and BCD is equal to the angle between the planes ACD and ABD (and similarly for other pairs of planes).
- The solid angles at each vertex have the same measure.

Proof 4.5. (Sketch) From Problem 4.4, there is a rigid motion that brings the equihedral tetrahedron to coordinates $A = (0, y, z), B = (x, 0, z), C = (x, y, 0), D = (0, 0, 0)$. From these coordinates, it is an easy to calculation to prove that Δ equihedral implies the other bullets.

- (2nd bullet implies first) Let $x = AD, y = BD, z = CD, a = BC, b = AC, c = AB$. Then $x+z+b = x+y+c = y+z+a = a+b+c$. This gives for example $z+b = y+c$, $x + b = y + a, x + z = a + c$. This gives $z = c$. This allows us to get $y = b, a = x$, proving that the tetrahedra is equihedral.
- (third bullet implies first) From the previous formula on the sum of pointiness of tetrahdra, the pointiness of each vertex is π. Thus, if we unravel the tetrahedra into a net, we get a triangle where ABC is the medial triangle. This immediately implies that the triangles are congruent.
- (fourth bullet implies first) UNFINISHED
- In a tetrahedron, the solid angles are the sum of the adjacent dihedral angles minus π. Thus, this bullet implies the first bullet follows from the previous part.

Equihedral tetrahedra turn out to be exceedingly interesting when discussing ant-paths. The following problems show us several examples of this correlation.

Problem 4.6. Prove that there are three pairwise intersecting simple closed ant-paths on a tetrahedron if and only if the tetrahedron is equihedral.

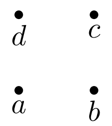
Problem 4.7. Two closed ant-paths on the surface of the tetrahedron are called *similar* if they intersect the edges of the tetrahedron in the exact same order. Prove that a tetrahedron has infinitely many non-similar closed ant-paths if and only if it is equihedral.

5 Ant-paths on cubes

Having thoroughly examined the case of both regular and equihedral tetrahedra, we turn to the case of the cube. More precisely, we are interested in which are the possible lengths of ant-paths on the surface of a cube. Unfortunately, when dealing with the cube, there is no statement analogous to problem 4.1. Therefore, when extending the claims about tetrahedra to cubes, we need to have a different approach when unrolling a cube onto the plane.

Problem 5.1. Let $ABCDA'B'C'D'$ be a cube and let α be the plane supporting the side $ABCD$. Rolling the cube over its edges onto α will give the integer-point lattice in the plane. Prove that it is not possible to label the vertices of this lattice by $a, b, c, d, a', b', c', d'$ so that the vertex A of the cube always lands at the point label a, vertex B at b, etc.

Proof 5.1. We must begin by labeling four points on the plane a, b, c, and d, such as in the diagram below.



Roll the cube to the right, then up, then to the left. Now the point labeled "d" above has vertex B of the cube. Therefore there is no labeling of the plane which is consistent with rolling the cube.

The last problem means that we will have to label the vertices on the plane depending on the ant-path we are examining at the moment. More precisely, to each closed ant-path, we will associate a straight planar segment of the same length. Then, we will examine which planar segments can be obtained from ant-paths, which will lead us towards a better understanding of ant-paths on the surface of the cube and of their lengths.

To construct a straight planar segment, we assume we have an ant-path $\mathcal{C}$ that intersects the edge AB and goes into the face $ABCD$ afterwards. Moreover, assume $\mathcal{C} \cap AB = X$ and $XA = d$. For the sake of simplicity we place the face $ABCD$ onto the unit square on the plane, and let X have coordinates $(d, 0)$. Then, we unroll the cube along $\mathcal{C}$. At some point, the point X will come back on the plane, thus marking the point X'.

Problem 5.2. Prove that the point X' will have the coordinates of the form $(d + m, n)$, for some positive integers $m, n \in \mathbb{Z}_{>0}$. Further, prove that the length of the planar line segment XX' is equal to the length of $\mathcal{C}$.

Proof 5.2. Since ant-paths of the cube cannot cross any vertex of the cube, the planar line segment corresponding to an ant-path must not touch any integer lattice points. This means that one can find a continuum of ant-paths that give the same sequence of unrolling moves; the corresponding line segments in the plane have the same slope but slightly different start points. This means we may assume d is irrational, since if it were rational we could simply consider nudging it to a nearby irrational number.

Consider what would happen if we kept unrolling the ant-path forever, obtaining a line in the plane rather than a segment. The sequence of unrolling steps (rolling right or up) repeats, so clearly this line, and therefore the original line segment, must have rational slope. X' must have one coordinate an integer and be distance d from a lattice point, meaning it is of the form $(m+d,n)$, $(m-d,n)$, $(m,n+d)$, or $(m,n-d)$ for some nonnegative integers m and n. But the slopes of the second through fourth possibilities are respectively $\frac{n}{m-2d}$, $\frac{n+d}{m-d}$, and $\frac{n-d}{m-d}$, which are irrational for irrational d. Therefore X' is of the form $(m+d,n)$.

Suppose now that $X' = (m+d,n)$. Then the ant-path (a broken line) consists of precisely $m+n$ segments whose lengths are precisely the lengths of $XX' \cap [a,a+1] \times [b,b+1]$ for some integers a and b. In other words, if you restrict the segment XX' to a the unit squares in the integer lattice, you will get $m+n$ smaller segments whose lengths match up exactly with the lengths of the segments of the original ant-path. Then the length of the ant-path is the same as the sum of the lengths of these planar segments, which is clearly just the length of XX'.

However, not every segment going from $(d,0)$ to $(d+m,n)$ corresponds to an ant-path.

Problem 5.3. Give an example showing the previous claim (i.e. find some d,m,n such that the segment from $(d,0)$ to $(d+m,n)$ cannot be produced from the ant-path using the above procedure).

Proof 5.3. Two easy counterexamples are $(d,m,n) = (1/2,3,2)$ and $(d,m,n) = (1/2,1,1)$. In the first case, the segment XX' intersects a lattice point, which is not possible for an ant-path. In the second case, the segment is too short to get a closed ant-path.

Therefore, our goal is to find which segments from $X = (d,0)$ to $X' = (d+m,n)$ correspond to actual closed ant-paths on the cube. There is one easy way to determine this for any d,m,n. Start by labeling the vertices of the unit square by a,b,c,d and by unrolling the cube over the segment XX'. When a vertex V of the cube falls onto the plane, label the point by v ($V \in \{A,B,C,D,A',B',C',D'\}$). If X' ends up on the horizontal edge whose left label is a and whose right label is b (the left label corresponding to the point (m,n), and the right on to the point $(m+1,n)$), then the cube have made a full rotation and we have an ant-path.

In general, given some fixed m,n, and $d \in [0,1]$, the labels of the points (m,n) and $(m+1,n)$ could depend on d. However, we have the following claim that eliminates this possibility in case the points (m,n) and $(m+1,n)$ were labeled a and b, as the following problem suggests.

Problem 5.4. If for given $m,n \in \mathbb{Z}_+$, the points (m,n) and $(m+1,n)$ get labels a,b for some d under the described procedure, they get the same labels for all but finitely many any $d \in [0,1]$.

Proof 5.4. TODO

The last problem tells us the fact whether the triple (m,n,d) corresponds to an ant-path does not really depend on d - if (m,n,d) comes from an ant-path for one value of d, it does so for almost any other value of d. Thus, one can define the pair (m,n) to be good if there is some d such that (m, n, d) corresponds to an ant-path. Moreover, as we have it is easy to see that m and n are enough to determine the length of the ant-path $\mathcal{C}$, because the length of the segment XX' is $XX' = \sqrt{m^2 + n^2}$.

This property helps us determine the lengths of all the ant-paths on a cube. However, before doing that, we will classify the simple - non-intersecting ant paths on a cube. To this end, we have two lemmas that provide a great reduction in terms of various m and n we have to consider.

Problem 5.5. Let $\mathcal{C}$ be a closed ant-path on the surface of the cube which has a self-intersection at a point A. Prove that the ant-path is perpendicular to itself at A.

Proof 5.5. It is either the case that A lies on some edge BC of the cube or that it lies on the interior of some face $WXYZ$. When unfolded, the ant-path becomes a straight line segment in the plane. The two times that point A is traversed by the ant-path correspond to two points in the plane, and when the cube is unrolled to these points, either the edge BC or the face $WXYZ$ touches the plane. But this edge or face can only be rotated in 90° increments in the plane since B, C, W, X, Y, and/or Z always land at integer points, so point A must traversed in perpendicular directions.

Problem 5.6. Let $\mathcal{C}$ be a closed ant-path on the surface of the cube which has an unfolding on the plane from $(d, 0)$ to $(m + d, n)$. Prove that if $m + n \geq 7$, the ant-path has a self-intersection.

Proof 5.6. Clearly $m + n$ is the total number of unrolling steps required to completely unroll the ant-path, and this is also the number of faces traversed by the ant-path. Therefore if $m + n \geq 7$, at least one face of the cube is traversed by the ant-path more than once. Let face $ABCD$ be traversed twice such that one of the traversals XY crosses edge AB at point X, and let the other traversal be WZ.

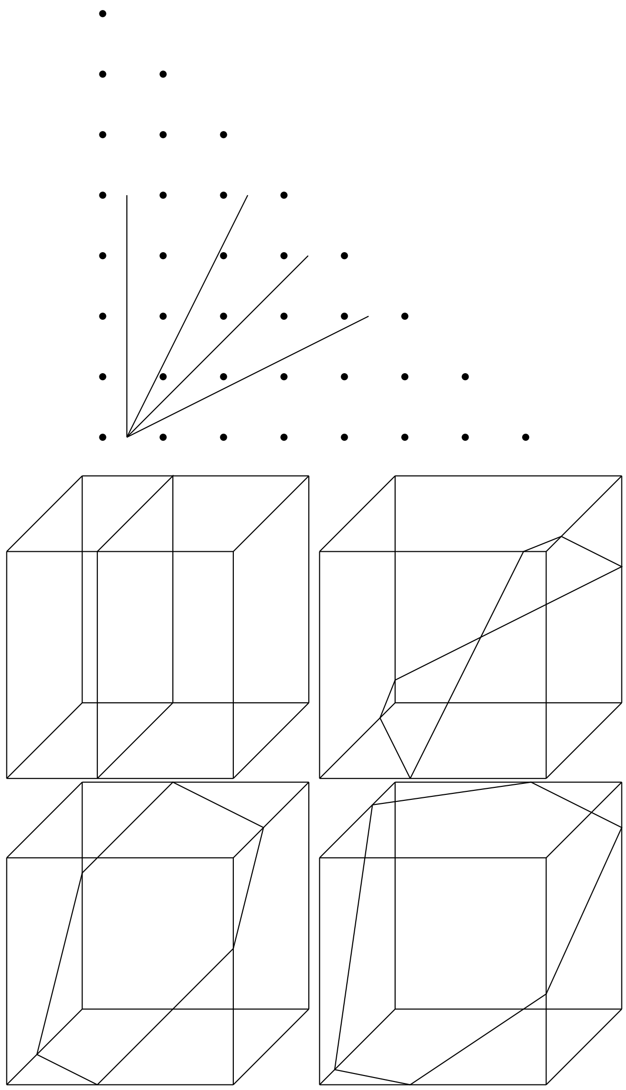
Now there are several cases to consider. If XY and WZ are perpendicular, then either they intersect on $ABCD$ or they both cross the same edge of $ABCD$, in which case the ant-path self-intersects on the other face incident to that edge. For example, if they both cross AB but do not intersect on $ABCD$, then the ant-path has a self-intersection on $ABD'C'$.

Otherwise, XY and WZ are parallel. There must be some choice of $ABCD$, XY, and WZ such that X, Y, W, and Z are on at least three unique edges of $ABCD$, since otherwise it would be impossible for XY and WZ to be part of the same broken line. Then without loss of generality $Y \in BC$, $Z \in CD$, and either $W \in AB$ or $W \in DA$ (we can either choose our labels A, B, C, D, X, Y, W, Z so this is the case, or mirror the cube and then choose labels so it is the case). Then it is clear that the ant-path has a self-intersection on face $BCA'D'$, face $CDB'A'$, or edge CA', depending on the exact position of XY and WZ.

Using the last problem, we can determine the lengths of all non-intersecting ant-paths on a cube.

Problem 5.7. Find the lengths of all non self-intersecting closed ant-paths on the surface of a cube. For each length you find, sketch an ant-path of that length on the surface of the cube.

Proof 5.7. There are four choices of (m, n) which give nonintersecting ant-paths: $(0, 4), (2, 4), (3, 3)$, and $(4, 2)$. These have lengths 4, $2\sqrt{5}$, $3\sqrt{2}$, and $2\sqrt{5}$, respectively. They are depicted below, unfolded and on the cube. Note there are only three unique lengths, but one is achievable in two ways.



Further, we will prove that we can find an ant-path corresponding to almost any direction in the plane.

Problem 5.8. Let m, n be two coprime non-negative integers (not both being zero). Then, there exists a unique integer $k \leq 4$ for which the pair (km, kn) corresponds to an ant-path (that does not repeat itself).

Hint: If (m, n) is not a good pair by itself, what are the labels of the points (m, n) and $(m + 1, n)$. Can you make any inference about the labels of $(2m, 2n)$ and $(2m, 2n + 1)$ based on this?

The statement of the last problem seems incomplete: how to determine k based on m, n? The following asks for any progress towards determining which k corresponds to which pairs m, n:

Problem 5.9. Prove that k always comes from $\{2, 3, 4\}$ in the last problem. Determine which values can k take depending on the parity of m, n.

Hint: Although there is a decisive answer to this question, this problem is intended to be open ended. Whatever conclusions you might have about how to determine k, make sure to write them down.

Finally, to outline the usefulness of this approach, we pose a computational question:

Problem 5.10. What are the five shortest lengths of closed, non-repeating ant-paths on the surface of the cube?

6 Connection to billiard trajectories

Let P be a given planar polygon. Then, we can naturally construct a family of polygons Q_ε from P in the following way: Q_ε will be the rectangular prism with the basis polygon congruent to P and height ε. For small ε, there is a natural correspondence between the ant-paths on Q_ε and the billiard trajectories on P.

Problem 6.1. Let P be a polygon which has a billiard trajectory not passing through any vertices of P. Then, prove that for all small enough ε there is an ant-path on the surface of Q_ε which passes through one of the bases. Conversely, prove that if for all small enough ε there is an ant-path on the surface of Q_ε which passes through one of the bases, then there is a billiard trajectory on P not passing through any vertices of P.

Team Number: ______________________________

PUMaC 2019 Power Round Cover Sheet

Remember that this sheet comes first in your stapled solutions. You should submit solutions for the problems in increasing order. Write on one side of the page only. The start of a solution to a problem should start on a new page. Please mark which questions for which you submitted a solution to help us keep track of your solutions.

Problem Number	Points	Solution written
1.1	10	y
1.2	20	y
1.3	20	y
1.4	15	y
2.1	20	y
2.2	20	y
2.3	20	y
2.4	20	Daniel
2.5	30	y
3.1	10	y
3.2	20	y
3.3	5	y
3.4	35	y
3.5	25	Aleksa
3.6	30	y
4.1	10	y
4.2	20	y
4.3	30	y
4.4	20	y
4.5	60	Aleksa
4.6	50	Aleksa
4.7	50	Aleksa
5.1	10	y
5.2	20	y
5.3	10	y
5.4	40	Aleksa
5.5	20	y
5.6	30	y
5.7	30	y
5.8	30	Daniel
5.9	60	Daniel
5.10	20	Daniel