P U M .: C

Team Round

The Team Round consists of 15 questions. Your team has **40 minutes** to complete the Team Round. Each problem is worth 5 points. At the conclusion of the test, in addition to providing answers to the questions, you will be asked to choose one of the fifteen questions below. If your answer to this chosen question is wrong, then your team automatically gets 0 bonus points. If you answered this question correctly, the number of bonus points you will receive is $\frac{5}{A}$, where A is the number of teams who also chose this problem and answered it correctly. Good luck!

1. Consider a 2021-by-2021 board of unit squares. For some integer k, we say the board is *tiled* by k-by-k squares if it is completely covered by (possibly overlapping) k-by-k squares with their corners on the corners of the unit squares. What is the largest integer k such that the minimum number of k-by-k squares needed to tile the 2021-by-2021 board is exactly equal to 100?

2. Gary is baking cakes, one at a time. However, Gary's not been having much success, and each failed cake will cause him to slowly lose his patience, until eventually he gives up. Initially, a failed cake has a probability of 0 of making him give up. Each cake has a $\frac{1}{2}$ of turning out well, with each cake independent of every other cake. If two consecutive cakes turn out well, the probability resets to 0 immediately after the second cake. On the other hand, if the cake fails, assuming that he doesn't give up at this cake, his probability of breaking on the next failed cake goes from p to $p + 0.5$. If the expected number of successful cakes Gary will bake until he gives up is $\frac{p}{q}$, for relatively prime p, q, find $p + q$.

3. Alice and Bob are playing a guessing game. Bob is thinking of a number n of the form 2^a3^b, where a and b are positive integers between 1 and 2020, inclusive. Each turn, Alice guess a number m, and Bob will tell her either $\gcd(m, n)$ or $\text{lcm}(m, n)$ (letting her know that he is saying that gcd or lcm), as well as whether any of the respective powers match up in their prime factorization. In particular, if $m = n$, Bob will let Alice know this, and the game is over. Determine the smallest number k so that Alice is always able to find n within k guesses, regardless of Bob's number or choice of revealing either the lcm, or the gcd .

4. Find the number of points $P \in \mathbb{Z}^2$ that satisfy the following two conditions:

   1) If Q is a point on the circle of radius $\sqrt{2020}$ centered at the origin such that the line $\overline{PQ}$ is tangent to the circle at Q, then $\overline{PQ}$ has integral length.

   2) The x-coordinate of P is 38.

5. Suppose two polygons may be glued together at an edge if and only if corresponding edges of the same length are made to coincide. A 3×4 rectangle is cut into n pieces by making straight line cuts. What is the minimum value of n so that it's possible to cut the pieces in such a way that they may be glued together two at a time into a polygon with perimeter at least 2021?

6. We say that a string of digits from 0 to 9 is *valid* if the following conditions hold: First, for $2 \leq k \leq 4$, no consecutive run of k digits sums to a multiple of 10. Second, between any two 0s, there are at least 3 other digits. Find the last four digits of the number of valid strings of length 2020.

7. Let X, Y, and Z be concentric circles with radii 1, 13, and 22, respectively. Draw points A, B, and C on X, Y, and Z, respectively, such that the area of triangle ABC is as large as possible. If the area of the triangle is Δ, find Δ^2.

8. Let there be a tiger, William, at the origin. William leaps 1 unit in a random direction, then leaps 2 units in a random direction, and so forth until he leaps 15 units in a random direction to celebrate PUMaC's 15th year.

   There exists a circle centered at the origin such that the probability that William is contained in the circle (assume William is a point) is exactly $\frac{1}{2}$ after the 15 leaps. The area of that circle can be written as $A\pi$. What is A?

9. Consider a regular 2020-gon circumscribed into a circle of radius 1. Given three vertices of this polygon such that they form an isosceles triangle, let X be the expected area of the isosceles triangle they create. X can be written as $\frac{1}{m\tan((2\pi)/n)}$ where m and n are integers. Compute $m+n$.

10. Let N be the number of sequences of positive integers greater than 1 where the product of all of the terms of the sequence is 12^{64}. If N can be expressed as $a(2^b)$, where a is an odd positive integer, determine b.

11. Three (not necessarily distinct) points in the plane which have integer coordinates between 1 and 2020, inclusive, are chosen uniformly at random. The probability that the area of the triangle with these three vertices is an integer is $\frac{a}{b}$ in lowest terms. If the three points are collinear, the area of the degenerate triangle is 0. Find $a+b$.

12. Given a sequence $a_0, a_1, a_2, \ldots, a_n$, let its *arithmetic approximant* be the arithmetic sequence $b_0, b_1, \ldots, b_n$ that minimizes the quantity $\sum_{i=0}^{n}(b_i - a_i)^2$, and denote this quantity the sequence's *anti-arithmeticity*. Denote the number of integer sequences whose arithmetic approximant is the sequence $4, 8, 12, 16$ and whose anti-arithmeticity is at most 20.

13. Will and Lucas are playing a game. Will claims that he has a polynomial f with integer coefficients in mind, but Lucas doesn't believe him. To see if Will is lying, Lucas asks him on minute i for the value of $f(i)$, starting from minute 1. If Will is telling the truth, he will report $f(i)$. Otherwise, he will randomly and uniformly pick a positive integer from the range $[1, (i+1)!]$. Now, Lucas is able to tell whether or not the values that Will has given are possible immediately, and will call out Will if this occurs. If Will is lying, say the probability that Will makes it to round 20 is $\frac{a}{b}$. If the prime factorization of b is $p_1^{e_1} \ldots p_k^{e_k}$, determine the sum $\sum_{i=1}^{k} e_i$.

14. Let N be the number of convex 27-gons up to rotation there are such that each side has length 1 and each angle is a multiple of $2\pi/81$. Find the remainder when N is divided by 23.

15. Suppose that f is a function $f : \mathbb{R}_{\geq 0} \to \mathbb{R}$ so that for all $x, y \in \mathbb{R}_{\geq 0}$ (nonnegative reals) we have that $f(x)+f(y) = f(x+y+xy)+f(x)f(y)$. Given that $f(\frac{3}{5}) = \frac{1}{2}$ and $f(1) = 3$, determine $\lfloor \log_2(-f(10^{2021} - 1)) \rfloor$.