P U M ∴ C

Live Round

1.1 [5] Let right triangle ABC have hypotenuse AC and $AB = 8$. Let D be the foot of the altitude from B to AC. If ABC has area 60, then the length of BD can be expressed in the form $\frac{a}{b}$, where a and b are relatively prime, positive integers. Find $a + b$.

1.2 [5] Zack climbs stairs at 2 meters per minute, Kevin climbs stairs at 7 meters per minute, and Rahul climbs stairs at 8 meters per minute. The Shanghai Tower is 632 meters tall. Zack begins climbing at noon, Kevin begins x minutes after him, and Rahul begins y minutes after Kevin. If they all arrive at the top at exactly the same time, compute $x + y$.

1.3 [5] Milan colors two squares, chosen uniformly at random without replacement, on a 3×3 grid. The expected number of vertices shared by the two squares is expressible as a/b where a, b are coprime positive integers. Find $a + b$.

2.1 [7] Let f be the cubic polynomial that passes through the points $(1, 30), (2, 15), (3, 10)$, and $(5, 6)$. Compute the product of the roots of f.

2.2 [7] In the top left corner of a grid with 100 rows and 100 columns is a ball. On each move the ball moves down one unit with probability 1/3 or right one unit with probability 2/3. After 99 moves, the ball will be in the nth column from the left, where $n = 1$ in the leftmost column. Find the expected value of n.

2.3 [7] Let σ be a permutation of the numbers $1, 2, 3, 4$. If

$$\sigma(a) \cdot \sigma^2(a) \cdot \sigma^3(a) \cdot \sigma^4(a) \equiv -1, \pmod 5$$

for all $a \in \{1, 2, 3, 4\}$, compute the number of possible σ.

Calculus 1 [5] If the shortest distance between a point on the curve $y = \sqrt{x^2 + 1}$ and the line $y = \frac{1}{2}x$ can be written in the form $\sqrt{\frac{a}{b}}$, where a, b are relatively prime integers, find $a + b$.

Estimation 1 [7] What is the number of permutations of $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ that have exactly 3 fixed points? (A fixed point is a number that is mapped to itself under the permutation).

You must give your answer as a nonnegative integer. If your answer is A and the correct answer is C, then your score will be $\lfloor 7.5 - \frac{|A-C|^{1.1}}{1500} \rfloor$.

Miscellaneous 1 [9] An infinite sequence of semicircles is constructed in the following way. The first semicircle has radius 2048. Then, each subsequent semicircle has diameter parallel to the previous, with endpoints on the previous semicircle's arc, and arc tangent to the previous semicircle's diameter. The limit of the distance of the nth semicircle's center from the first's may be written as $a\sqrt{b} - c$, where b is square-free and a, b, c are positive integers. Find $a + b + c$.

4.1 [9] Let a, b, and c be real numbers that satisfy the following equations: $a+b+c=0$ and $abc=2019$. Compute $a^3+b^3+c^3$.

4.2 [9] Consider a triangle where the sum of the three side lengths is equal to the product of the three side lengths. If the circumcircle has 25 times the area of the incircle, the distance between the incenter and the circumcenter can be expressed in the form $\frac{\sqrt{x}}{y}$, for integers x and y, with x square-free. Find $x+y$.

4.3 [9] Find the number of different possible values of the number of parts in which we can cut a circle with 2018 distinct lines (assuming that all the lines cut the circle in its interior).

5.1 [12] What is the sum of all positive integers n with at most three digits that satisfy $n=(a+b)\cdot(b+c)$ when n is written in base 10 as $\underline{a}\ \underline{b}\ \underline{c}$? Note: The integer n can have leading zeroes.

5.2 [12] Let k be the number of nonintersecting paths a King can take on a 6×6 square board from one corner to the opposite corner such that the number of steps the King is from its starting point never decreases. Compute $k \mod 23$. (Note that a King can move to any square that shares an edge or a vertex with its current square)

5.3 [12] The sequence a_n satisfies $a_1=1$, $a_2=3$ and, for $n\geq 2$, $a_n=\frac{1}{n+2}\left(\sum_{j=1}^{n+1} a_j - 1\right)$. Compute $\left\lceil \log_2 \frac{a_{2020}}{a_{2018}} \right\rceil$.

Calculus 2 [9] Consider the triangle with vertices $(0,0),(1,0)$, and $(0,1)$. Let A, B, and C denote the distances from a given point to each of the three vertices. Denote the distance from the point that minimizes $A+B+C$ to the point that minimizes $A^2+B^2+C^2$ by d. If d is written as $\frac{\sqrt{a}-\sqrt{b}}{c}$ where a and b are square free, find $a+b+c$.

Estimation 2 [12] Let $\phi(x)$ denote the number of positive integers less than x that are relatively prime to x. Estimate the value of $\sum_{x=1}^{1000} \phi(x)$.

You must give your answer as a nonnegative integer. If your answer is A and the correct answer is C, then your score will be $\lfloor 12.5 \min((\frac{A}{C})^8, (\frac{C}{A})^8) \rfloor$.

Miscellaneous [15] Compute

$$\left\lfloor \sum_{n=0}^{49} \sin\left(\frac{\pi n}{100}\right) \right\rfloor.$$

7.1 [15] Find the sum of all positive integers k such that there exists a positive integer a such that $7k^2 = a^3 + a! + 2767$.

7.2 [15] If a, b, c are positive reals such that $abc = 64$ and $3a^2 + 2b^3 + c^6 = 384$, compute maximum value of $a + b + c$.

7.3 [15] Three fair twenty-sided dice are rolled, and then arranged in decreasing order. The expected value of the largest die can be written in the form $\frac{p}{q}$ where p and q are relatively prime positive integers. Find $p + q$.

8.1 [18] A (b, k)-*palindrome code* is a sequence of k integers between 0 and $b - 1$, inclusive, that reads the same forwards as backwards. Note that a (b, k)-palindrome code can be interpreted as a base-b integer, if one ignores the initial zeros. A positive integer n is *reliable* if there exist at least two distinct pairs of positive integers b and k, both greater than 1, such that the average of all (b, k)-palindrome codes (interpreted as base-b integers) is equal to n. Find the sum of the three smallest reliable numbers.

8.2 [18] Define $S, T \subset \{1, \ldots, 2016\}$ such that S consists of all such integers that are divisible by 3 and T consists of the rest. Compute

$$\sum_{s \in S} \binom{2019}{s} - \frac{1}{2} \sum_{t \in T} \binom{2019}{t}.$$

8.3 [18] In $\triangle ABC$, let $\angle CAB = 45\,\text{deg}$, and $|AB| = \sqrt{2}, |AC| = 6$. Let M be the midpoint of side BC. The line AM intersects the circumcircle of $\triangle ABC$ at P. The circle centered at M with radius MP intersects the circumcircle of ABC again at $Q \neq P$. Suppose the tangent to the circumcircle of $\triangle ABC$ at B intersects AQ at T. Find TC^2.

Calculus 3 [15] Let p_n denote the nth prime number. If the limit

$$\lim_{n\to\infty}\prod_{k=1}^{n}\left(\frac{p_n}{p_k}\right)^{\frac{p_n}{n(p_n+p_k)}}$$

is expressed as $e^{\frac{\pi^a}{b}}$ for integers a and b, find $a+b$.

Estimation 3 [18] Randomly choose points from $[0,35]\times[0,35]$ uniformly. If N is the expected number of points we must choose before two points are within a distance 1 of each other, find the nearest integer to N.

You must give your answer as a nonnegative integer. If your answer is A and the correct answer is C, then your score will be $\lfloor 18.5\min(\left(\frac{A}{C}\right)^9,\left(\frac{C}{A}\right)^9)\rfloor$.

Miscellaneous 3 [20] Two countries each have three cities connected together by paved roads. To ensure better travel between the cities, the countries decide to hire some people to pave the roads between each city. Each person will walk from city to city only along unpaved roads and pave them along the way. Each of the nine roads connecting two cities of different countries has a $\frac{1}{2}$ probability of being already paved before the pavers were hired. If the expected minimum of pavers that the countries will need to hire to pave all the roads is $\frac{m}{n}$, find $m+n$.