Number Theory A

1. Shaq sees the numbers 1 through 2017 written on a chalkboard. He repeatedly chooses three numbers, erases them, and writes one plus their median. (For instance, if he erased $-2, -1, 0$ he would replace them with 0.) If M is the maximum possible final value remaining on the board, and if m is the minimum, compute $M - m$.

2. The sequence of positive integers $a_1, a_2, \ldots$ has the property that $\gcd(a_m, a_n) > 1$ if and only if $|m - n| = 1$. Find the sum of the four smallest possible values of a_2.

3. Define the *bigness* of a rectangular prism to be the sum of its volume, its surface area, and the lengths of all of its edges. Find the least integer N for which there exists a rectangular prism with integer side lengths and bigness N and another one with integer side lengths and bigness $N + 1$.

4. For any integer $n \geq 2$, let b_n be the least positive integer such that, for any integer N, m divides N whenever m divides the digit sum of N written in base b_n, for $2 \leq m \leq n$. Find the integer nearest to b_{36}/b_{25}.

5. Let $p(n) = n^4 - 6n^2 - 160$. If a_n is the least odd prime dividing $q(n) = |p(n-30) \cdot p(n+30)|$, find $\sum_{n=1}^{2017} a_n$. ($a_n = 3$ if $q(n) = 0$.)

6. Find the least positive integer N such that the only values of n for which $1 + N \cdot 2^n$ is prime are multiples of 12.

7. Compute the number of ordered pairs of integers (a, b), where $0 \leq a < 17$ and $0 \leq b < 17$, such that $y^2 \equiv x^3 + ax + b \pmod{17}$ has an even number of solutions (x, y), where $0 \leq x < 17$ and $0 \leq y < 17$ are integers.

8. Find the minimum value attained by $\sum_{m=1}^{100} \gcd(M - m, 400)$ for M an integer in the range $[1746, 2017]$.