PUMaC 2023 Power Round: An Introduction to Algebraic Geometry

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Rules and Reminders

1. Your solutions should be turned in by **5PM Thursday, November 16th, EDT**. You will submit the solutions through Gradescope. The instructions describing how to log into Gradescope will be sent to the coaches. The deadline for submission is clearly visible on the Gradescope site once you enroll in the course.

   Please make sure you submit your work in time. **No late submissions will be accepted.** Please do not submit your work using email or in any other way. If you have questions about Gradescope, please post them on Piazza.

   You may either typeset the solutions in LaTeX or write them by hand. We strongly encourage you to typeset the solutions. This way, the proofs end up being more clear and the chances are you will not lose points there. Moreover, you might want to use some of the LaTeX resources listed in point 2.

   In case your solutions are handwritten, the cover sheet (the last page of this document) should be the first page of your submission. In case you typeset your solutions, please take a look at the Solutions Template we posted and make sure to make the cover sheet the first page of your submission.

   Each page should have on it the **team number** (not team name) and **problem number**. This number can be found by logging in to the coach portal and selecting the corresponding team. Solutions to problems may span multiple pages. Please put them in order when submitting your solutions.

2. You are encouraged, but not required, to use LaTeX to write your solutions. If you submit your power round electronically, **you may submit several times, but only your final submission will be graded** (moreover, you may not submit any work after the deadline). The last version of the power round solutions that we receive from your team will be graded. Moreover, **you must submit a PDF**. No other file type will be graded. For those new and interested in LaTeX, check out Overleaf as well as its online guides. If you do not know the specific command for a math symbol, check out Detexify or TeX.StackExchange.

3. Do not include identifying information aside from your team number in your solutions.

4. Please collate the solutions in order in your submission. Each problem should start on a new page (there is a point deduction for not following this formatting).

5. On any problem, you may use without proof any result that is stated earlier in the test, as well as any problem from earlier in the test, even if it is a problem that your team has not solved. These are the **only** results you may use. In particular, to solve a problem, you may not cite the subsequent ones. You may not cite parts of your proof of other problems: if you wish to use a lemma in multiple problems, please reproduce it in each one.

6. When a problem asks you to "find", "find with proof," "show," "prove," "demonstrate," or "ascertain" a result, a formal proof is expected, in which you justify each step you take, either by using a method from earlier or by *proving* that *everything* you do is correct. When a problem instead uses the word "explain," an informal explanation suffices. When a problem instead uses the word "sketch" or "draw" a clearly marked diagram is expected.

7. All problems are numbered as "Problem x.y.z" where x.y is the subsection number and z is the the number of the problem within the subsection. Each problem's point distribution can be found in the cover sheet.

8. **You may NOT use any references, such as books or electronic resources, unless otherwise specified. You may NOT use computer programs, calculators, or any other computational aids.**

9. Teams whose members use English as a foreign language may use dictionaries for reference.

10. **Communication with humans outside your team of 8 students about the content of these problems is prohibited.**

11. There are two places where you may ask questions about the test. The first is Piazza. Please ask your coach for instructions to access our Piazza forum. On Piazza, you may ask any question **so long as it does not give away any part of your solution to any problem**. If you ask a question on Piazza, all other teams will be able to see it. If such a question reveals all or part of your solution to a power round question, your team's power round score will be penalized severely. For any questions you have that might reveal part of your solution, or if you are not sure if your question is appropriate for Piazza, please email us at pumac@math.princeton.edu. We will email coaches with important clarifications that are posted on Piazza.

Introduction and Advice

In this power round, we state and prove a special case of the "cubic surface theorem". This theorem states that every smooth cubic surface in $\mathbb{P}^3$ contains exactly 27 lines (if you don't understand what any of this means at this point, don't worry!). This is a rather striking result, and we hope you will find it as intricate and beautiful as we do.

A large part of the difficulty in this power round will arise from the many different perspectives that one needs to understand the material and tackle the problems. For example, one can many times solve problems by looking at things from an algebraic standpoint, working explicitly with equations, as well as with more complicated algebraic structures. On the other hand, there is a vital geometric component, as you will see. These two perspectives marry to give a unique and exquisite sub-field of mathematics; this is known as algebraic geometry.

Here is some further advice with regard to the Power Round:

- **Read the text of every problem!** Many important ideas are included in problems and may be referenced later on. In addition, some of the theorems you are asked to prove are useful or even necessary for later problems.

- **Make sure you understand the definitions**. A lot of the definitions are not easy to grasp; don't worry if it takes you a while to fully understand them. If you don't, then you will not be able to do the problems. Feel free to ask clarifying questions about the definitions on Piazza (or email us).

- **Don't make stuff up**: on problems that ask for proofs, you will receive more points if you demonstrate legitimate and correct intuition than if you fabricate something that *looks* rigorous just for the sake of having "rigor."

- **Check Piazza often!** Clarifications will be posted there, and if you have a question it is possible that it has already been asked and answered in a Piazza thread (and if not, you can ask it, assuming it does not reveal any part of your solution to a question). **If in doubt about whether a question is appropriate for Piazza, please email us at pumac@math.princeton.edu.**

- **Don't cheat**: as stated in Rules and Reminders, you may **NOT** use any references such as books or electronic resources. If you do cheat, you will be disqualified and banned from PUMaC, your school may be disqualified, and relevant external institutions may be notified of any misconduct.

Good luck, and have fun!

– Kaivalya Kulkarni, Colby Riley

We would like to acknowledge and thank many individuals and organizations for their support; without their help, this Power Round (and the entire competition) could not exist. Please refer to the solutions of the power round for full acknowledgments and references.

Contents

Notation

- $\forall$: for all. *Ex.:* $\forall x \in \{1,2,3\}$ *means "for all* x *in the set* $\{1,2,3\}$*"*
- $A \subset B$: proper subset. *Ex.:* $\{1,2\} \subset \{1,2,3\}$*, but* $\{1,2\} \not\subset \{1,2\}$
- $A \subseteq B$: subset, possibly improper. *ex.:* $\{1\}, \{1,2\} \subseteq \{1,2\}$
- $f : x \mapsto y$: f maps x to y. *Ex.: if* $f(n) = n-3$ *then* $f : 20 \mapsto 17$ *and* $f : n \mapsto n-3$ *are both true.*
- $f(C)$: for a function $f : A \to B$ and subset $C \subseteq A$, the set of elements of the form $f(c)$, for $c \in C$.
- $\{x \in S : C(x)\}$: the set of all x in the set S satisfying the condition $C(x)$. *Ex.:* $\{n \in \mathbb{N} : \sqrt{n} \in \mathbb{N}\}$ *is the set of perfect squares.*
- $\mathbb{N}$: the natural numbers, $\{1,2,3,\dots\}$.
- $\mathbb{Z}$: the integers.
- $\mathbb{Q}$: the rational numbers.
- $\mathbb{R}$: the real numbers.
- $\mathbb{C}$: the complex numbers.
- $|S|$: the cardinality of set S.
- $\forall x = a + bi \in \mathbb{C}$, $|x| = \sqrt{a^2 + b^2}$.

1 Preliminaries from Linear Algebra and Topology

The 2 sections on linear algebra will not contain problems, but it is highly recommended to read through the material; this will make you much more comfortable in the rest of the power round.

The sections on Topology **will** contain problems!

1.1 Complex Vector Spaces

A natural setting for much of the power round will be in Complex Vector Spaces, which will allow us to apply the structure of $\mathbb{C}$ to other spaces! We will see many of these spaces throughout the power round.

Definition 1.1.1. A complex vector space V is a set equipped with two operations, addition from $V \times V \to V$ and scalar multiplication from $\mathbb{C} \times V \to V$. Elements of V are called *vectors*. Addition satisfies the following properties:

1. (Associativity) For all $u, v, w \in V$, $u + (v + w) = (u + v) + w$.

2. (Commutativity) For all $v, w \in V$, $v + w = w + v$.

3. (Existence of zero) There exists an element known as the zero vector, denoted 0 such that $v + 0 = v$ for all $v \in V$.

4. (Existence of inverse) For each $v \in V$, there exists an inverse element, denoted $-v$, such that $v + (-v) = 0$.

Scalar Multiplication satisfies the following properties:

1. (Associativity) For all $\alpha, \beta \in \mathbb{C}$ and $v \in V$, $\alpha(\beta v) = (\alpha\beta)v$.

2. (Multiplicative Identity) For each $v \in V$, $1v = v$.

3. (Distributivity I) For all $\alpha, \beta \in \mathbb{C}$, we have that $(\alpha + \beta)v = \alpha v + \beta v$.

4. (Distributivity II) For all $v, w \in V$ and all $\lambda \in \mathbb{C}$, we have $\lambda(v + w) = \lambda v + \lambda w$.

Example. The set of all n tuples of complex numbers forms a complex vector space with component wise addition and component wise scalar multiplication. We will denote this space $\mathbb{C}^n$.

Example. Consider $\mathbb{C}[x]$, the set of all polynomials in 1 variable with coefficients in the the complex numbers $\mathbb{C}$ (e.g. $(1+i)x^3 - 4x + i \in \mathbb{C}[x]$). This forms a complex vector space, with standard polynomial addition and scalar multiplication. This is an example of a vector space that is "infinite dimensional" - we will come back to this notion later.
We introduce several fundamental notions:

Definition 1.1.2. Let V be a (complex) vector space. A **subspace** $U \subset V$ is a subset of a vector space V satisfying the following properties:

1. for all $u, v \in U$, $u + v \in U$.

2. for all $\alpha \in \mathbb{C}$ and $u \in U$, $\alpha u \in U$.

Every subspace of a vector space V is itself a vector space.

Definition 1.1.3. A set of vectors $\{v_1, \cdots, v_k\}$ is said to be **linearly dependent** if there exists scalars $\alpha_1, \cdots, \alpha_k$, not all zero, such that

$$\alpha_1 v_1 + ... + \alpha_k v_k = 0 \tag{1}$$

If there does not exist such a collection of scalars, then the set is **linearly independent**.

Example. In $\mathbb{C}^2$, the set $\{(1, i), (3i, i), (3 + i, 0)\}$ is linearly dependent, as

$$1(1, i) + -1(3i, i) + i(3 + i, 0) = (0, 0) = 0 \tag{2}$$

Meanwhile, in $\mathbb{C}[x]$, the set $\{ix^2, x, 90 + i + x\}$ is linearly independent.

Definition 1.1.4. The **span** of a set of vectors $\{v_1, ..., v_k\}$ is the set $\{\alpha_1 v_1 + ... + \alpha_k v_k : \alpha_1, ..., \alpha_k \in \mathbb{C}\}$. That is, it is all possible **linear combinations** of the vectors $\{v_1, \cdots, v_k\}$.

Definition 1.1.5. A **basis** for a vector space V is a linearly independent set of vectors which spans the whole vector space V.

The **dimension** of a vector space is the size of the basis. It is an incredible fact that this is actually well-defined - two bases of the same vector space *must* be the same size. (One last thing: the vector space consisting only of the 0 vector is considered to have dimension 0).

The above conversation has so far concerned only finite bases; these form finite dimensional vector spaces. We can introduce similar notions of linear dependence, span and basis for infinite sets of vectors.

Definition 1.1.6. An infinite set of vectors A is **linearly dependent** if there exists a linearly dependent finite subset of A. They are **linearly independent** if no finite linearly dependent subset of A exists. A **spans** a space V if every element in V can be written as a finite linear combination of elements in A.

And basis is defined in the same way as before. One can check that $\mathbb{C}[x]$ from before is an infinite dimensional vector space with a basis of $\{1, x, x^2, x^3, ...\}$.

1.2 Linear Transformations

We now develop the notion of maps between vector spaces. Let V and U be (complex) vector spaces, and let T be a map from V to U (at the moment, this is just a map of sets). We want T to respect the operations of our vector spaces, vector addition and scalar multiplication:

1. If we add two vectors v_1, v_2 in V and then apply T to this sum $(T(v_1 + v_2))$, we want this to be the same if we first applied T to v_1 and v_2 and then add the two vectors in U $(T(v_1) + T(v_2))$.

2. If we multiply two vectors v_1, v_2 in V by a scalar $c \in \mathbb{C}$, and then apply T ($T(cv_1)$), we want this to be the same if we first applied T to v_1 and then multiplied by c ($cT(v_1)$).

We are led to the following:

Definition 1.2.1. A **linear transformation** between a vector space V and a vector space U is a mapping $T : V \to U$ such that for all $v_1, v_2 \in V$, $c \in \mathbb{C}$,

$$T(v_1 + v_2) = T(v_1) + T(v_2) \tag{3}$$

$$T(cv_1) = cT(v_1) \tag{4}$$

Linear transformations have a particularly nice description in terms of bases. If $\{v_1, ..., v_k\}$ is a basis for a finite dimensional vector space V, then any linear map $T : V \to W$ is completely determined by where it sends basis vectors. This is because any vector $v \in V$ may be expressed as a linear combination of $\{v_1, \cdots, v_k\}$, and repeatedly applying (3) and (4) we see that $T(v)$ is completely determined by the values $\{T(v_1), \cdots, T(v_k)\}$.
Furthermore, specifying a map of sets $\{v_1, \cdots, v_k\} \to W$ *uniquely* defines a linear map of vectors space $T : V \to W$; can you see why?
Now, consider the vector spaces U and W, each with a basis $\{u_1, ..., u_j\}$ and $\{w_1, ..., w_k\}$. From the previous paragraph, a linear transformation $T : U \to W$ is determined by its values on $\{u_1, \cdots, u_j\}$, and we can write $T(u_i)$ as a linear combination of $\{w_1, ..., w_k\}$ for each $1 \leq i \leq j$: $T(u_i) = c_{i1}w_1 + \cdots + c_{ik}w_k$, $c_{i1}, \cdots, c_{ik} \in \mathbb{C}$. We can make a grid out of these elements in the following way:

$$\begin{pmatrix} c_{11} & \dots & c_{j1} \\ \vdots & \ddots & \vdots \\ c_{1k} & \dots & c_{jk} \end{pmatrix} \tag{5}$$

You may notice that this looks exactly like a matrix. In fact, it is! Remember that $\{v_1, ..., v_j\}$ is our basis for V. Then we can write how a matrix transforms a vector as

$$\begin{pmatrix} c_{11} & \dots & c_{j1} \\ \vdots & \ddots & \vdots \\ c_{1k} & \dots & c_{jk} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_j \end{pmatrix} = \begin{pmatrix} \alpha_1 c_{11} + ... + \alpha_j c_{j1} \\ \vdots \\ \alpha_1 c_{1k} + ... + \alpha_j c_{jk} \end{pmatrix} \tag{6}$$

Here, the 1-row column on the left represents the vector $\alpha_1 v_1 + ... + \alpha_j v_j$, and the output represents $(\alpha_1 c_{11} + ... + \alpha_j c_{j1})u_1 + ... + (\alpha_1 c_{1k} + ... + \alpha_j c_{jk})u_k$. Thus, to each linear transformation $T : U \to W$ we may associate a matrix, which is dependent on a choice of bases for both U and V. This is an extremely useful idea for computational purposes. Many times, we will blur the distinction between a linear transformation and its matrix; in this case, the choice of bases will be implied implicitly.
One important point:

Theorem 1.2.2. *Let V, W and U be vector spaces with bases $\{v_1, \cdots, v_k\}$, $\{w_1, \cdots, w_j\}$, and $\{u_1, \cdots, u_n\}$, respectively. Let $T : V \to W$ and $S : W \to U$ be linear transformations. Then the matrix corresponding to the composite $S \circ T$ is given by AB, where A is the matrix of S with respect to the given bases, and B is the matrix of T with respect to the given bases.*

Definition 1.2.3. When discussing the vector spaces $\mathbb{C}^n$, there is an almost implicit basis used called the **standard basis**. This consists of the n elements $\{(1, 0, ..., 0), (0, 1, 0, ..., 0), ..., (0, ..., 0, 1)\}$. You can check that this forms a basis and that therefore $\mathbb{C}^n$ is an n-dimensional vector space.

Given a linear transformation $T : V \to U$, the image $T(V)$ is a subspace of U. The dimension of $T(V)$ is extremely important, and is called the **rank** of T.

Example.

1. The mapping $T : \mathbb{C}^3 \to \mathbb{C}^1$ such that $T((a, b, c)) = a + b + c$ has rank 1.
2. The mapping $T : \mathbb{C}^1 \to \mathbb{C}^3$ such that $T(a) = (a, 0, 0)$ has rank 1.
3. The zero mapping $T : V \to W$ such that $T(v) = 0$ has rank 0.

The vector space $T(V)$ is a subspace of U. Meanwhile, the *kernel* $\ker(T)$, the set of all vectors v such that $T(v) = 0$, is a subspace of V.

Theorem 1.2.4. *(Rank-Nullity) Let $T : V \to W$ be a linear mapping of vector spaces. Then,*

$$\dim(\ker(T)) + \dim(T(V)) = \dim(V) \tag{7}$$

The rank of T can actually be found just by looking at a matrix A which represents T (in any basis!). First notice that the columns of T may be interpreted as vectors in W. Call this subspace spanned by these vectors $\mathrm{col}(A)$. Then, we have that $\mathrm{rank}(T) = \dim \mathrm{col}(A)$. Furthermore, each row of A may be naturally interpreted as a vector in V; call the subspace spanned by these vectors $\mathrm{row}(A)$. Then we have the $\mathrm{rank}(T) = \dim \mathrm{col}(A) = \dim \mathrm{row}(A)$. (Aside: Given any matrix A, we may define $\mathrm{rank}(A)$ to be either $\dim \mathrm{col}(A)$ or $\dim \mathrm{row}(A)$, as these quantities are equal.)

Example. In the standard basis for $\mathbb{C}^3$, the following matrices have rank 2:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{8}$$

$$\begin{pmatrix} 1 & 0 & 1+i \\ i & 0 & -1+i \\ 1+3i & 1 & -2+4i \end{pmatrix} \tag{9}$$

Thus, their corresponding linear transformations have rank 2, and hence the dimension of their respective kernels is 1.

One last fact:

Theorem 1.2.5. *Let A be a square matrix of size $n \times n$, and recall the quantity det(A) (the determinant of A). Then, we have that* $\det(A) \neq 0$ *if and only if* $\operatorname{rank}(A) = n$*, or equivalently,* $\dim(\ker(A)) = 0$.

1.3 Topological Spaces

We now give some topological background that will be needed later.

Definition 1.3.1. A topological space is a set X along with a collection $\mathcal{E}$ of subsets (denoted $(X, \mathcal{E})$), that satisfy the following properties:

1. The empty set $\emptyset$ and the set X are contained in the collection $\mathcal{E}$.
2. Let $\{E_\alpha\}$ be an arbitrary collection of elements in $\mathcal{E}$. Then the union $\bigcup E_\alpha$ is contained in the collection $\mathcal{E}$.
3. Let $E_1, \cdots, E_k$ be a finite collection of elements in $\mathcal{E}$. Then the intersection $\bigcap_{i=1}^k E_i$ is contained in the collection $\mathcal{E}$.

A subset A of X is said to be open if $A \in \mathcal{E}$, and the collection $\mathcal{E}$ is said to define a topology on X.

Example. Let X be any set, and let $\mathcal{E} = \{X, \emptyset\}$. This defines a topology on X: clearly $\mathcal{E}$ is closed under union and finite intersection, and by construction the empty set and X are contained in $\mathcal{E}$. This is known as the indiscrete topology on X.

Problem 1.3.1. Let X be any set, and let $\mathcal{E} = P(X)$, the power set of X. Show that $\mathcal{E}$ defines a topology on X. This is known as the discrete topology on X. (The Power set $P(X)$ is defined as the set of all subsets of X. For example, if $X = \{1, 2\}$, then $P(X) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$.)

Problem 1.3.2. Let X be the set $\{a, b, c\}$ and let $\mathcal{E} = \{\{a\}, \{b\}, \{c\}, \{a, b\}, \{a, b, c\}, \emptyset\}$. Does $\mathcal{E}$ define a topology on X? Why or why not?

Problem 1.3.3. List all the possible topologies on $X = \{a, b, c\}$ (up to relabeling).

Sometimes, it is more convenient to talk about sets whose *complement* is open. That is, in a space X, a subset $C \subset X$ is said to be **closed** if $C^c = \{x \in X : x \notin C\}$ is open. In fact, we can also define a topology by specifying its closed sets.

We now introduce an important concept.

Definition 1.3.2. Let X be a topological space, and let $Y \subset X$ be a subset. A point $p \in X$ is said to be a limit point of Y if for each open set U containing p, $U \cap Y$ contains a point in Y other than p. Furthermore, if L_Y is the union of all limit points of Y, then $L_Y \cup Y$ is said to be the closure of Y in X, denoted $\bar{Y}$.

Problem 1.3.4. Let X be a topological space, and let $Y \subset X$ be any subset. Show that $\bar{Y}$ is a closed subset of X.

Problem 1.3.5. Let X be a topological space. Show that $Y \subset X$ is closed if and only if $Y = \bar{Y}$.

Now we will introduce an important topology: the **classical topology on** $\mathbb{C}$.

For any point $z = a+bi \in \mathbb{C}$ and a number $\epsilon > 0 \in \mathbb{R}$, define the open ball $B_\epsilon(z)$ around z to be the set of all points $\{p \in \mathbb{C} : |p - z| < \epsilon\}$.

Definition 1.3.3. A set $X \subset \mathbb{C}$ is open if $X = \bigcup_{i \in J} A_i$, where each $A_i = B_{\epsilon_i}(z_i)$ for some ϵ_i and z_i and where J is an arbitrary indexing set, or if X is the empty set. That is, a nonempty set is open if it is the arbitrary union of open balls in $\mathbb{C}$.

The indexing set may be finite, infinite or even uncountable. It is actually not immediate that this definition satisfies the axioms for a topology.[1]

Problem 1.3.6. Show that the above definition satisfies the axiom for a topology. In particular, show that the intersection of two open sets is open. (Hint: take an open ball around each point).

Problem 1.3.7. Show that for any open set in $\mathbb{C}$, it is actually the union of a *countable* number of open balls, that is, that our indexing set J from earlier may be taken to be countable. (Hint: Use the fact (you don't need to prove this) that $\mathbb{Q}[i] = \{a + bi \in \mathbb{C} \mid (a, b) \in \mathbb{Q}^2\}$ is dense in $\mathbb{C}$; that is, for any $x \in \mathbb{C}$ and $\epsilon > 0$, there exists $q \in \mathbb{Q}[i]$ so that $|x - q| < \epsilon$).

We now define the notion of **bases** for a topology.

Definition 1.3.4. Let $(X, \mathcal{E})$ be a topological space. A **basis** for $\mathcal{E}$ is a collection of open sets $\mathcal{B}$ such that every $U \in \mathcal{E}$ may be expressed as the union of a subcollection of elements in $\mathcal{B}$.

With this definition in mind, in the above example we see that open balls form a basis for the topology of $\mathbb{C}$. We now describe a few more important definitions and theorems that will be useful later.

It is many times the case where we have two spaces $X \subset Y$, and we want to describe a topology on X in terms of that on Y.

Definition 1.3.5. For $X \subset Y$ and a given topology on Y, the **subspace topology on** X is given as:

$U \subset X$ is open in X if and only if it is of the form $U = V \cap X$ for $V \subset Y$ open in Y.

Problem 1.3.8. Verify that the subspace topology is indeed a topology.

[1] for a set X to be countable means that there is a bijective function from X to the natural numbers $\mathbb{N}$. To be uncountable means that X is infinite and there is no such bijection. For example, $\{5\}$ is finite, $\mathbb{Z}$ and $\mathbb{Q}$ are countable, and $\mathbb{R}$ and $\mathbb{C}$ are uncountable.

We now introduce the product topology.

Definition 1.3.6. Let $\{(X_i, \mathcal{E}_i)\}$, $1 \leq i \leq n$, be a finite collection of topological spaces. Then, we may consider the Cartesian product ΠX_i. This is naturally a topological space: a subset $A \in \Pi X_i$ is said to be open if it may be expressed as the union of sets of the form $U_1 \times \cdots, \times U_n$ where each $U_i \in \mathcal{E}_i$. This topology is known as the product topology on the space ΠX_i.[2]

Example. Consider the space $\mathbb{C}^2$ consisting of all pairs of complex numbers (a, b). Then $\mathbb{C}^2$ is the product $\mathbb{C} \times \mathbb{C}$, and hence inherits a topological structure from $\mathbb{C}$ in the form of the product topology.

Problem 1.3.9. Notice that there is a canonical map $\phi : \mathbb{C} \to \mathbb{R}^2$ given by $a + bi \to (a, b)$. Furthermore, $\mathbb{R}^2$ may be equipped with the product topology from $\mathbb{R}$ (We say that a subset $U \subset \mathbb{R}$ is open if it may be written as the union of open intervals). Show that a subset $U \subset \mathbb{C}$ is open if and only if the set $\phi(U)$ is open in $\mathbb{R}^2$.

There is another, slightly more complicated way to form new spaces. It is called the quotient topology, and it can be thought of as an operation where one "glues" points of a space together to form a new space.

We first introduce the notion of an equivalence relation:

Definition 1.3.7. Let S be a (nonempty) set. An equivalence relation on S is a subset Σ of the Cartesian Product $S \times S$ satisfying the following conditions:

1. (Reflexive Property) for each $a \in S$, $(a, a) \in \Sigma$.

2. (Symmetric Property) if $(a, b) \in \Sigma$, then so is (b, a).

3. (Transitive Property) if $(a, b) \in \Sigma$ and $(b, c) \in \Sigma$, then so is (a, c).

Example. Let S be a set, and let $S_1, \cdots, S_k$ be a collection of pairwise disjoint subsets such that $\bigcup_{i=1}^k S_i = S$. Consider the subset Σ of the Cartesian Product $S \times S$ consisting of pairs (a, b) where both a and b lie in S_i, for the same i. Then, Σ is an equivalence relation: for each $a \in S$, a lies in some S_i as the union $\bigcup_{i=1}^k S_i$ is S. Thus, (a, a) is contained in Σ. Similarly, if $(a, b) \in \Sigma$, then this implies that a and b lie in some S_i, for the same i. In turn, this implies that $(b, a) \in \Sigma$. A similar argument shows the transitive property: $(a, b) \in \Sigma$ and $(b, c) \in \Sigma$, then so is (a, c). It turns out that all equivalence relations in turn define a pairwise disjoint collection of sets like S_i, which from now on we will call equivalence classes.

[2] For products of infintely many spaces, the situation is more complicated; however, as we will not be dealing with infinite products, we do not discuss this case here.

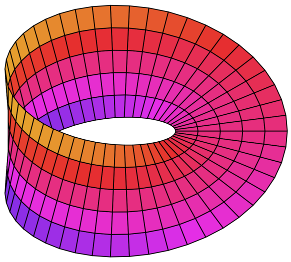

Figure 1: The Moebius Strip

> **Problem 1.3.10.** Prove the transitive property of equivalence relations for the above example.

Another way to notate equivalence relations is as follows: given an equivalence relation Σ on a set S, we write $a \sim b$ if and only if $(a, b) \in \Sigma$. We will use this notation for the rest of the power round.

Now take any topological space X with an equivalence relation defined on it. Now let the new space $X/_\sim$ be defined as the set of all equivalence classes on X. For any $p \in X$, let $[p] \in X/_\sim$ be the equivalence class that p belongs to: this gives a map q from X to $X/_\sim$. Then we define the open sets in Y to be precisely as follows:

Definition 1.3.8. A set U is open in $X/_\sim$ if and only if $\{x \in X : [x] \in U\}$ is open in X.

Example. Let D (the unit disk) be the set of points $D = \{z \in \mathbb{C} : |z| \leq 1\}$, and introduce an equivalence relation as follows: if $|z| < 1$, then $z \sim w$ if and only if $z = w$. If z, w are such that $|z| = |w| = 1$, then $z \sim w$. Then $D/_\sim$ can be thought of as a sphere: this can be visualized as taking a filled-in disk and gluing the circular edge all into one point. [insert image].

Example. Let I^2 be the unit square in the complex plane (that is, $\{a + bi : a \in [0, 1], b \in [0, 1]\}$, and let $z, w \in I^2$ be equivalent if $z = 0 + bi$ and $w = 1 + (1 - b)i$, for $0 \leq b \leq 1$ (or vice versa). Then $I^2/_\sim$ is the Möbius strip, a square whose two edges are identified with a half twist. See the figure above.

Before we move on, there are a few properties that may be desirable for some topological spaces but not others. For instance, in $\mathbb{C}$'s classical topology, for any two points $z, w \in \mathbb{C}$, one can make small enough open balls around z and w such that the balls do not intersect. Formally, we have this definition:

Definition 1.3.9. A space S is **Hausdorff** if for any two $x, y \in S$, if $x \neq y$, then there exists open sets $x \in U_x, y \in U_y$ such that $U_x \cap U_y = \emptyset$.

Example. $\mathbb{C}$ is Hausdorff. Given any two points in $\mathbb{C}$, one can always draw two open balls around the points that do not intersect.

Problem 1.3.11. For any set X, is the indiscrete topology on X Hausdorff? What about the discrete topology? (Hint: You will need to make cases.)

Problem 1.3.12. Let X be a Hausdorff space, and let $Y \subset X$ have the subspace topology: is Y Hausdorff? Similarly, if $Y \subset X$ has the subspace topology from X, and Y is Hausdorff, must X also be Hausdorff? If X and Y are two spaces, and they are both Hausdorff, is $X \times Y$ Hausdorff? What if only X is Hausdorff?

There is one last property that will be relevant to our power round, namely, Noetherian topological spaces.

Definition 1.3.10. A space X is Noetherian if for every descending sequence

$$X_1 \supset X_2 \supset X_3 \supset \ldots$$

of closed sets, there is some m such that $X_m = X_{m+1} = X_{m+2} = \ldots$. In other words, there cannot exist a infinite descending chain of closed subsets.

Example. $\mathbb{C}$ is *not* Noetherian. For example, we can take the sequence of closed balls $X_n = \{z : |z| \leq \frac{1}{n}\}$, which descends infinitely.

Problem 1.3.13. Prove that if a space is both Hausdorff and Noetherian, then it must be discrete. (Hint: the finite case is true even if we only assume Hausdorff).

1.4 Continuity

Definition 1.4.1. Let X and Y be topological spaces. A continuous morphism (or map, mapping, function, etc.) is a map $f : X \to Y$ such that the inverse image of every open set $O \subset Y$ is an open subset of X.

Problem 1.4.1. Let $f : X \to Y$ be a mapping of topological spaces. Show that f is continuous if and only if the inverse image of every closed set is closed.

Example. For the Cartesian product $X \times Y$, the projection mapping $p : X \times Y \to X$ is continuous, since for any open set $U \subset X$, $p^{-1}(U) = U \times Y$, which is an open set in $X \times Y$.

Example. For a topological space X and an equivalence relation $\sim$, the mapping $q : X \to X/_\sim$ that sends $x \in X$ to its equivalence class $[x] \in X/_\sim$ is continuous since, by definition, a set $U \subset X/_\sim$ is open only if $q^{-1}(U)$ is open.

Example. Here is an example of a map that is not continuous. Consider the map $f : \mathbb{C} \to \mathbb{C}$ that sends z to 1 if $|z| > 0$, and sends z to 0 otherwise. The map is not continuous since the inverse image of the open ball $B_{\frac{1}{2}}(0)$ of radius one half around 0 is the one point set $\{0\}$, which is not open in the classical topology on $\mathbb{C}$. (Note that the inverse image of $B_{\frac{1}{2}}(1)$ is in fact $\mathbb{C} - \{0\}$, which *is* open).

In practice, given a map f between topological spaces X and Y, checking whether $f^{-1}(V)$ is open for each open set $V \subset Y$ can be quite tedious. Thankfully, there are simpler ways to check whether a map is continuous. Namely, given a basis B_Y of the topology on Y, it suffices to check that $f^{-1}(W)$ is open for each $W \in B_Y$.

Problem 1.4.2. Verify that the above assertion is actually true; i.e. given a map $f : X \to Y$ such that $f^{-1}(W)$ is open in X for each $W \in B_Y$ (B_Y is as above), show that f is continuous.

Problem 1.4.3. For the following maps, decide (with proof) whether the map is continuous or not:

- The identity map $\mathrm{id} : X \to X$ that sends $x \in X$ to itself.
- The constant map $c : X \to \{x_0\}$ that sends everything to the one point set $\{x_0\}$.
- The map from $\mathbb{C}$ with the classical topology to $\mathbb{C}$ with the discrete topology which is the identity on each element.

Problem 1.4.4. Let X, Y, and Z be topological spaces, and suppose $f : X \to Y$ and $g : Y \to Z$ are continuous maps. Show that the composition $g \circ f : X \to Z$ is continuous.

A continuous map is said to be a homeomorphism between two spaces if it is a bijection and has an inverse which is also continuous. If a homeomorphism exists between two spaces, they are said to be homeomorphic.

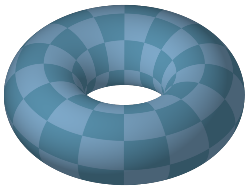

Figure 2: A picture of T from problem 1.4.8

Problem 1.4.5. Let X, Y, and Z be topological spaces, and suppose $f : X \to Y$ and $g : Y \to Z$ are homeomorphisms. Show that the composition $g \circ f : X \to Z$ is a homeomorphism.

Problem 1.4.6. Give an example of two spaces X, Y and a map $f : X \to Y$ such that f is continuous and bijective, but its inverse is not continuous.

Problem 1.4.7. Given an example of a homeomorphism between the open unit interval $(0, 1) \subset \mathbb{R}$ to the whole real line $\mathbb{R}$. (Hint: You may assume trigonometric functions and inverse trignometric functions are continuous on their respective domains).

The classic joke among Topologists is that they cannot tell the difference between a donut and a coffee cup, since they are homeomorphic. This Topologist does not find it so funny.

Problem 1.4.8. Introduce an equivalence relation on the complex plane as follows: $a + bi \sim c + di$ iff $c - a \in \mathbb{Z}$ and $d - b \in \mathbb{Z}$ and denote $\mathbb{C}/_\sim$ as T. Then let $S^1 \subset \mathbb{C}$ be the set $S^1 = \{z : |z| = 1\}$, with the subspace topology. Show that T is homeomorphic to $S^1 \times S^1$.

Problem 1.4.9. Let $A \subset B$ with the subspace topology. A retraction $r : B \to A$ is a continuous function from B to A such that for all $a \in A$, then $r(a) = a$. Take as given that there is no retraction from the unit disk $D \subset \mathbb{C}$ to the unit circle $S^1 \subset \mathbb{C}$. Using this, prove that for any continuous function f from the unit disk to itself, there exists some $x \in D$ such that $f(x) = x$. (Hint: Argue by contradiction, and assume that there exists a function $f : D \to D$ with no fixed points. How can you construct a retraction from $D \to S^1$ from such a function f?)

2 Algebraic Geometry

2.1 Affine and Projective Spaces

We start with a definition fundamental to algebraic geometry.

Definition 2.1.1. Consider the set of all n-tuples of complex numbers,

$$\{(a_1, \cdots, a_n) \, |a_1, \cdots, a_n \in \mathbb{C}\}. \tag{10}$$

This is known as n-dimensional complex affine space, denoted $\mathbb{A}^n(\mathbb{C})$ or $\mathbb{A}^n$. Notice that $\mathbb{A}^n$ also has the structure of a vector space.

Affine space has a natural topological structure, namely the product topology from $\mathbb{C}$. In opposition to affine space, there is also projective space, denoted $\mathbb{P}^n(\mathbb{C})$ or simply $\mathbb{P}^n$.
We now define complex projective space:

Definition 2.1.2. Consider the set $\mathbb{A}^{n+1} - (0, \cdots, 0)$, where each element is represented by an $n+1$-tuple of complex numbers $(a_0, \cdots, a_n)$, and at least one a_i is nonzero for some i. Define an equivalence relation as follows: $(a_0, \cdots, a_n) \sim (b_0, \cdots, b_n)$ if there exists a nonzero $\lambda \in \mathbb{C}$ such that

$$(a_0, \cdots, a_n) = (\lambda b_0, \cdots, \lambda b_n) \tag{11}$$

Then, $\mathbb{P}^n$ is set of equivalence classes under this equivalence relation. Geometrically, one may think of $\mathbb{P}^n$ as the space of lines in $\mathbb{A}^{n+1}$.

Problem 2.1.1. Check that the equivalence relation defined above is actually an equivalence relation.

$\mathbb{P}^n$ has a natural topological structure, namely the quotient topology from $\mathbb{A}^{n+1} - (0, \cdots, 0)$. Points in $\mathbb{P}^n$ are denoted as $[X_0, \cdots, X_n]$, where the bracket notation is used to indicate that $[X_0, \cdots, X_n]$ represents the equivalence class of points equivalent to $(X_0, \cdots, X_n) \in \mathbb{A}^{n+1} - \{0\}$ under the equivalence relation defined above.

2.2 Affine and Projective Varieties

We introduce the next fundamental idea in algebraic geometry.

Definition 2.2.1. Let f be a polynomial in n variables, say $x_1, \cdots, x_n$. The (complex) *zero locus* of f is defined to be the collection of points $(a_1, \cdots, a_n) \in \mathbb{A}^n$ such that

$$f(a_1, \cdots, a_n) = 0. \tag{12}$$

Example. Consider the polynomial $f(x_1, x_2) = x_1 + x_2$. The zero locus of f consists of points $(a_1, a_2) \in \mathbb{A}^2$ such that $a_1 = -a_2$.

Definition 2.2.2. An affine variety in $\mathbb{A}^n$ is the intersection of the zero loci of a collection of polynomials in n variables $x_1, \cdots, x_n$.

Example. The zero locus of the polynomial $f(x_1, x_2) = x_1 + x_2$ is an affine variety. More generally, given any polynomial f in variables $x_1, \cdots, x_n$, the zero locus of f is an affine variety in $\mathbb{A}^n$.

Example. Consider the polynomial f defined above, $f(x_1, x_2, x_3) = x_1 + x_2$, as well as the polynomial $g(x_1, x_2, x_3) = x_1 + x_2 + x_3$. Then the intersection of the of the zero loci on these two polynomials is an affine variety in $\mathbb{A}^3$. We now give a more explicit description of this variety.
Every point (a_1, a_2, a_3) on this variety satisfies two conditions: $a_1 + a_2 = 0$, and $a_1 + a_2 + a_3 = 0$. These two conditions imply that $a_3 = 0$. Thus, we find that this variety consists of all points $(a_1, a_2, a_3) \in \mathbb{A}^3$ of the form $(a, -a, 0)$, $a \in \mathbb{C}$. This is an example of a special kind of variety, known as a linear variety.

Every affine variety has a natural topological structure, namely the subspace topology from $\mathbb{A}^n$. This is known as the *classical* topology on V (as opposed to the *Zariski* topology, which will be discussed later).

Example. Let $A = \{\alpha_1, \cdots, \alpha_k\}$ be any finite collection of points in $\mathbb{A}^1(\mathbb{C})$. We claim that A is in fact an affine variety. In particular, notice that A is exactly the zero locus of the polynomial $P(x_1) = (x_1 - \alpha_1) \cdots (x_1 - \alpha_k)$.

Problem 2.2.1. Generalize the preceding example in the following way. Let $A = \{\alpha_1, \cdots, \alpha_k\}$ be any finite collection of points in $\mathbb{A}^n(\mathbb{C})$. Show that A is an affine variety in $\mathbb{A}^n(\mathbb{C})$. (Hint: Show that $\{\alpha_1\}$ is an affine variety first. Then use induction!)

One last generalization:

Problem 2.2.2. Let $V_1, \cdots V_k$ be a finite collection of affine varieties in $\mathbb{A}^n$. Show that the union $\bigcup_{i=1}^k V_i$ is an affine variety.

We now consider the case of *projective* varieties. To define projective variety, we first define several preliminary notions:

Definition 2.2.3. A monomial in n variables $x_1, \cdots, x_n$ is a polynomial of the form $x_1^{n_1} \cdots x_k^{n_k}$.

The *degree* of a monomial is the sum $n_1 + \cdots + n_k$.

Definition 2.2.4. Let $P(X_0, \cdots, X_n)$ be a polynomial. We may write P (uniquely up to reordering) as the sum of finitely many monomials: $P = \sum \alpha_i X_0^{i_0} \cdots X_n^{i_n}$. The *degree* of P is the maximum of the degrees of each monomial.

Example. Consider the polynomial $P(X_1, X_2, X_3) = X_1 + X_1X_2 + X_3^4X_2X_1$. Then the degrees of the monomials that make up P are 1, 2, and 6, respectively. The maximum of these numbers is 6; hence the degree of P is 6.

$P = \sum \alpha_i X_0^{i_0} \cdots X_n^{i_n}$ is said to be homogeneous if each monomial $\alpha_i X_0^{i_0} \cdots X_n^{i_n}$ has the same degree. In this case, $\deg(P)$ is defined to be the degree of any monomial in the above sum.

Example. The polynomial $P(X_0, X_1) = X_0 + X_1$ is homogeneous, as P is the sum of two monomials each of degree 1.

Example. The polynomial $P(X_0, X_1, X_2) = X_0X_1(X_0 + X_1^2)$ is not homogeneous. To see this, we write P as a sum of monomials: $P = X_0^2X_1 + X_0X_1^3$, and note that one monomial has degree 3, while the other has degree 4.

Problem 2.2.3. Consider the following polynomials:

1. $P(X_0, X_1) = X_0(X_0 + X_1)$
2. $P(X_0, X_1, X_2) = X_0^3 + X_1^4 + X_0X_1X_2$
3. $P(X_0, X_1, X_2, X_3) = X_0^{1003}$

Which are homogeneous, and which ones are not? Why?

Problem 2.2.4. Let $P_1, \cdots P_k$ be a collection of homogeneous polynomials in variables $X_0, \cdots, X_n$. Show that the product $\Pi_{i=1}^k P_i$ is homogeneous.

Homogeneous polynomials, as opposed to ordinary polynomials, have one key feature that allows us to think of their zero loci as living in projective space as opposed to affine space. In particular, if P is a homogeneous polynomial in variables $(X_0, \cdots, X_n)$, and $(\alpha_0, \cdots, \alpha_n)$ is a point in the zero locus of P, then for any nonzero $\lambda \in \mathbb{C}$,

$$P(\lambda\alpha_0, \cdots, \lambda\alpha_n) = \lambda^d P(\alpha_0, \cdots, \alpha_n) = 0 \tag{13}$$

where d is the degree of P. In other words, whenever P vanishes at a point $(\alpha_0, \cdots, \alpha_n)$, we must have that P vanishes on the entire line $(\lambda\alpha_0, \cdots, \lambda\alpha_n)$. It therefore makes sense to think of the zero locus of a homogeneous polynomial as collection of points in the space of *lines* in affine space - hence, zero loci of homogeneous polynomials define certain subsets of projective space. We are led to the definition of projective variety:

Definition 2.2.5. A projective variety in $\mathbb{P}^n$ is the intersection of zero loci of a collection of homogeneous polynomials in $n+1$ variables $X_0, \cdots, X_n$.

As in the affine case, every projective variety V has a natural topological structure, namely the subspace topology from $\mathbb{P}^n$. This is also known as the classical topology on V.

Problem 2.2.5. Show that any finite collection of points in $\mathbb{P}^n$ is a projective variety. (Hint: Mimic the proof in the affine case)

2.3 The Zariski Topology

We now introduce a topology onto both affine space $\mathbb{A}^n$ and projective space $\mathbb{P}^n$, that differs from the classical topology in several key ways.

Definition 2.3.1. Consider the following collection of subsets $\mathfrak{U}$ of affine space $\mathbb{A}^n$: for each $U \in \mathfrak{U}$, U^c (complement of U in $\mathbb{A}^n$) is the zero locus of a collection of polynomials; e.g. is an affine variety. Then this collection $\mathfrak{U}$ of open sets defines a topology on $\mathbb{A}^n$, known as the *Zariski* topology on $\mathbb{A}^n$. A subset of $\mathbb{A}^n$ is said to be *Zariski open* if it belongs to $\mathfrak{U}$. Likewise, a subset whose complement belongs to $\mathfrak{U}$ is said to be *Zariski closed.* (That is, closed sets are exactly the affine varieties).

The axioms of a topological space can be checked; it suffices to check that $\mathbb{A}^n$ and $\emptyset$ are closed, finite unions of closed sets are closed, and arbitrary intersections of closed sets are closed. The first condition follows easily, the second follows from an exercise in the previous section, and the third also follows immediately.

Example. Consider the affine line $\mathbb{A}^1$ with a point removed, say $x_0 \in \mathbb{A}^1$. Then $\mathbb{A}^1 - \{x_0\}$ is Zariski open; the set $\{x_0\}$ is an affine variety. In general, subsets of the form $\mathbb{A}^n - \{x_0, \cdots, x_n\}$ are Zariski open, as the complements of such subsets are affine varieties.

Notice that every affine variety V inherits a subspace topology from the Zariski topology on $\mathbb{A}^n$. We call this topology the Zariski topology on V.

Definition 2.3.2. Let U be a subset of $\mathbb{A}^n$. U is said to be a quasi-affine variety if there exists an affine variety V such that U is a Zariski-open subset of V.

We now define the Zariski topology on projective varieties.

Definition 2.3.3. Consider the collection of subsets of projective space $\mathbb{P}^n$, each subset U satisfying the following property: U^c (complement of U in $\mathbb{P}^n$) is the zero locus of a collection of homogeneous polynomials; e.g. is an projective variety. Then this collection $\mathfrak{U}$ of open sets defines a topology on $\mathbb{P}^n$, known as the *Zariski* topology on $\mathbb{P}^n$. A subset of $\mathbb{P}^n$ is said to be *Zariski open* if it belongs to $\mathfrak{U}$. Likewise, a subset whose complement belongs to $\mathfrak{U}$ is said to be *Zariski closed.*

Just like the affine case, one can readily check that the Zariski topology does in fact satisfy the axioms for a topological space. There is a notion of "quasi"-projective as well:

Definition 2.3.4. Let $U \subset \mathbb{P}^n$. U is said to be a quasi-projective variety if there exists a projective variety V such that $U \subset V$ and U is open in the Zariski topology on V.

We state a quick fact about the Zariski topology (without proof):

Theorem 2.3.5. *When considered with the Zariski topology, $\mathbb{A}^n$ and $\mathbb{P}^n$ are Noetherian topological spaces.*

Problem 2.3.1. Let V be a variety (i.e. either an affine or projective variety). Show that V may be described as the common zero locus of finitely many polynomials.

Problem 2.3.2. Prove that $\mathbb{A}^n$ with the Zariski topology is not Hausdorff.

There is a relationship between the classical topology and the Zariski topology. Namely, every Zariski open subset (of $\mathbb{A}^n$ or $\mathbb{P}^n$) is open in the classical topology:

Problem 2.3.3. Let V be an (affine or projective) variety in $\mathbb{A}^n$ (resp. $\mathbb{P}^n$). Show that V is closed in the classical topology on $\mathbb{A}^n$ (resp. $\mathbb{P}^n$). Conclude that every Zariski open set is classically open.
(Hint: You may use the fact that polynomials are continuous maps $\mathbb{A}^n \to \mathbb{A}^1$ in the classical topologies).

It is natural to ask if the converse is true. Thankfully, it is not (otherwise, Zariski topology and the classical topology would be the same!). The following exercise demonstrates this.

Problem 2.3.4. Consider the affine line $\mathbb{A}^1$, and let

$$X = \{x + iy \mid x = 0,\ y \in \mathbb{R}\}. \tag{14}$$

Show that X is closed in the classical topology, but not in the Zariski topology.

Projective space $\mathbb{P}^n$ is naturally "covered" by copies of $\mathbb{A}^n$: for each $0 \leq i \leq n$, consider the subset $U_i = \{[X_0, \cdots, X_n] \mid X_i \neq 0\} \subset \mathbb{P}^n$. There is a natural map $\phi_i : U_i \to \mathbb{A}^n$ given by $[X_0, \cdots, X_n] \to (\frac{X_0}{X_i}, \cdots, \frac{X_{i-1}}{X_i}, \frac{X_{i+1}}{X_i}, \cdots \frac{X_n}{X_i})$. Notice that the assumption $X_i \neq 0$ allows ϕ_i to be well-defined.

Problem 2.3.5. Check that the map ϕ_i is well defined, and a bijection for each $0 \leq i \leq n$.

This map has more structure than it may seem at the moment; it is actually a homeomorphism in the Zariski topology! To see this, we first describe a process which one can use to "go between" affine and projective varieties.
Let $V \subset \mathbb{P}^n$ be an projective variety, and consider the intersection $V \cap U_i$ for each $0 \leq i \leq n$, and recall the map $\phi_i : U_i \to \mathbb{A}^n$.

Problem 2.3.6. Let V be as above. Show that $\phi_i(V \cap U_i) \subset \mathbb{A}^n$ is in fact an affine variety. This is known as an affine chart for V.

Now, let $V \subset \mathbb{A}^n$ be an affine variety. Then, for each $0 \leq i \leq n$, consider the map $\phi_V^{-1} : V \to U_i$ defined by restricting the map $\phi_i^{-1} : \mathbb{A}^n \to U_i$ to V. Let $\bar{V}$ denote the closure of the image of ϕ_V^{-1}. Then, it follows (by definition) that $\bar{V}$ is a projective variety; this is known as a projective closure of the affine variety V.

Problem 2.3.7. Let V and $\bar{V}$ be as above. Show that $\bar{V} \cap U_i = \phi_V^{-1}(V)$.

Problem 2.3.8. Using the previous two exercises, show that the map $\phi_i : U_i \to \mathbb{A}^n$ defined above is a homeomorphism (in the Zariski topology).

2.4 First Examples of Varieties

Definition 2.4.1. Let $V \subset \mathbb{P}^n$ be a projective variety. A *subvariety* of V is a projective variety $W \subset \mathbb{P}^n$ such that $W \subset V$.

Example. Consider the projective variety V defined by the polynomial $P(X_1, X_2) = (X_1 + X_2)(X_1^2 + X_2^2)$. Then, V contains the varieties V_1 and V_2 defined by the equations $P_1(X_1, X_2) = X_1 + X_2$ and $P_2(X_1, X_2) = X_1^2 + X_2^2$, respectively.

Problem 2.4.1. Let $P(X_0, \cdots, X_n) = \Pi_{i=1}^{i=k} P_i(X_0, \cdots, X_n)$, where each P_i is homogeneous. Show that the variety V defined by P contains the varieties V_i as subvarieties, where each V_i is defined by P_i.

Definition 2.4.2. Let $V \subset \mathbb{P}^n$ (resp. $\mathbb{A}^n$) be a projective (resp. affine) variety. V is said to be *irreducible* if there do not exist distinct, proper subvarieties V_1 and V_2 such that $V_1 \cup V_2 = V$.

Problem 2.4.2. Consider the affine variety $V \subset \mathbb{A}^3$ given by the equations

$$x_1^2 - x_2 x_3 = 0 \tag{15}$$

and

$$x_1 x_3 - x_1 = 0. \tag{16}$$

Show that V is not irreducible, and that V may be written as the union of 3 distinct, proper subvarieties.

Problem 2.4.3. Consider the projective variety $V \subset \mathbb{P}^3$ defined by the equations

$$P_0(X_0, X_1, X_2, X_3) = X_0 X_2 - X_1^2 \tag{17}$$

and

$$P_1(X_0, X_1, X_2, X_3) = X_0 X_3 - X_1 X_2. \tag{18}$$

Show that V is not irreducible.

Definition 2.4.3. Let $V \subset \mathbb{P}^n$ be a variety given by a single homogeneous polynomial P, and let d be the degree of P. Then, V is said to be a hypersurface of degree d in $\mathbb{P}^n$.

Example. Let $V \subset \mathbb{P}^n$ be a projective variety given by the equation

$$P(X_0, \cdots, X_N) = \alpha_0 X_0 + \cdots \alpha_n X_n, \tag{19}$$

where the α_i's are scalars in $\mathbb{C}$, not all zero. Then V is a hypersurface of degree 1. Such a hypersurface is known as a *hyperplane*.

We know introduce an important class of varieties, known as linear varieties.

Definition 2.4.4. Let $V \subset \mathbb{P}^n$ be a projective variety. V is said to be *linear* if there is a collection of hyperplanes $\{H_i\}$ such that $V = \bigcap H_i$.

The Noetherian structure of $\mathbb{P}^n$ (in the Zariski topology) implies that every linear variety V is in fact the intersection of finitely many hyperplanes, say $H_0, \cdots H_{k-1}$. Suppose that H_i is defined by the equation $\alpha_{i0}X_0 + \cdots + \alpha_{in}X_N = 0$. This defines a $k \times (n+1)$ matrix of coefficients (α_{ij}) where $0 \leq i \leq k-1$ and $0 \leq i \leq n$. Thinking of this matrix as a linear map $T : \mathbb{A}^{n+1} \to \mathbb{A}^k$, we see that the projection of the kernel of this map to $\mathbb{P}^n$ is precisely V.

Problem 2.4.4. Prove rigorously the above assertion; i.e. show that V is $\ker(T)$ modulo scalar equivalence.

Now, we give some definitions pertaining to linear varieties.

Definition 2.4.5. Let $V \subset \mathbb{P}^n$ be a linear variety. Then, the dimension of V is said to be $\dim(\ker(T)) - 1$, where T is the linear transformation associated to *any* matrix of coefficients for V. In particular, it does not matter which equations for V we choose, we will always get a well-defined notion of dimension.

Problem 2.4.5. Let $V \subset \mathbb{P}^n$ be a projective variety that consists of a single point. Show that $\dim(V) = 0$ (Implicitly, you must show that V is linear!). Conversly, let $V \subset \mathbb{P}^n$ be a linear variety such that $\dim(V) = 0$. Show that V consists of a single point.

There is a way to define the dimension of an arbitrary projective (or affine) variety; however doing so would take us a bit outside the scope of this power round.

Definition 2.4.6. Let $\{H_i\}_{i=1}^{n-1}$ be a collection of $n-1$ hyperplanes in $\mathbb{P}^n$, such that the coefficient matrix has rank $n-1$. Then, the intersection $\bigcap H_i$ is said to be a line.

By Rank-Nullity Theorem, this is equivalent to saying that the linear transformation T associated to the matrix of coefficients satisfies $\dim(\ker(T)) = 2$, and hence $\dim(V) = 1$.

Example. Let l_1 and l_2 be two distinct lines in $\mathbb{P}^2$. Then, we claim that the intersection of l_1 and l_2 is exactly one point. In particular, let l_1 be defined by the equation

$$\alpha_0 X_0 + \alpha_1 X_1 + \alpha_2 X_2 = 0 \tag{20}$$

and let l_2 be defined by the equation

$$\beta_0 X_0 + \beta_1 X_1 + \beta_2 X_2 = 0 \tag{21}$$

Consider the matrix of coefficients

$$\begin{pmatrix} \alpha_0 & \alpha_1 & \alpha_2 \\ \beta_0 & \beta_1 & \beta_2 \end{pmatrix}$$

Then, as l_1 and l_2 are distinct, the rows above the above matrix are linearly independent, and hence the matrix has rank 2. Thus, by the Rank-Nullity Theorem, we find that the kernel has dimension 1, and the variety defined by the two equation given above above is a linear variety of dimension 0. Hence, it is a point.

Problem 2.4.6. Consider the Fermat surface $V \subset \mathbb{P}^3$ given by the equation

$$P(X_0, X_1, X_2, X_3) = X_0^3 + X_1^3 + X_2^3 + X_3^3. \tag{22}$$

Show that the Fermat surface contains a line.

Problem 2.4.7. Give an example of two lines l_1 and l_2 in $\mathbb{P}^3$ such that l_1 and l_2 do not intersect.

2.5 Regular Maps and Morphisms of Varieties

As usual, we first deal with the affine case. From here on out, we will work exclusively in the Zariski topology.

Definition 2.5.1. Let $V \subset \mathbb{A}^n$ be an affine variety, and let U be an open subset. A regular function on U is a map $\phi : U \to \mathbb{A}^1$ with the following property: For each point $p \in U$, there is an open neighbourhood $U_p \subset U$ of p such that restricted to U_p, ϕ can be expressed as a ratio $\frac{g}{h}$, where g and h are polynomials in n variables, and $h(x) \neq 0$ for all $x \in U_p$.

Informally, the above definition is saying that regular functions locally are rational functions. Hopefully some examples will clarify things a bit.

Example. Consider the affine line $\mathbb{A}^1$, and the map $\phi : \mathbb{A}^1 \to \mathbb{A}^1$ given by $x \to x+5$. Then this function is regular; for each point $x \in \mathbb{A}^1$, notice that $\mathbb{A}^1$ is an open neighborhood of x, and ϕ may be expressed as a ratio $\frac{g}{h}$ on $\mathbb{A}^1$ with $g(x) = x+5$ and $h(x) = 1$.

Example. Consider the open set $U = \mathbb{A}^1 - \{0\}$ in $\mathbb{A}^1$. Then, the function $f(x) = \frac{1}{x}$ is a regular function on U; for each $p \in U$, U is an open neighborhood of p, and f may be expressed as a ratio $\frac{g}{h}$ on U with $g(x) = 1$ and $h(x) = x$.

Problem 2.5.1. Let f be a polynomial in n variables. Show that as a function $f : \mathbb{A}^n \to \mathbb{A}^1$, f is a regular function.

Amazingly, although we have defined regular functions to only *locally* resemble rational functions, regular functions have a very rigid global structure. In particular, we have the following:

Theorem 2.5.2. *Let U be an open subset of $\mathbb{A}^n$, and suppose that $U = \{x \in \mathbb{A}^n \mid f(x) \neq 0\}$ for some polynomial f in n variables. Then every regular function on U takes the form $\frac{g}{f^k}$, where g is a polynomial and k is a non-negative integer. In particular, let f be a regular function on $\mathbb{A}^n$. Then f is a polynomial in n variables.*

Unfortunately, we are not in a position to present a proof of this incredible fact. But, we can already prove some wonderful things with this theorem.

Problem 2.5.2. Let f be a regular function on $\mathbb{A}^2 - (0,0)$. Show that f *extends* to a regular function on $\mathbb{A}^2$; i.e. there exists a regular function $\hat{f}$ on $\mathbb{A}^2$ such that $\hat{f} = f$ on $\mathbb{A}^2 - (0,0)$.

We now define regular maps of (quasi-) affine varieties.

Definition 2.5.3. Let $V \subset \mathbb{A}^n$ be a (quasi-) affine variety, and let $W \subset \mathbb{A}^m$ be a (quasi-) affine variety. Then we may represent any map $\phi : V \to W$ as taking $(x_1, \cdots, x_n) \to (\phi_1(x_1, \cdots, x_n), \cdots, \phi_m(x_1, \cdots, x_n))$. ϕ is said to be regular map if each coordinate function ϕ_i is a regular function.

Example. Let $\phi : \mathbb{A}^n \to \mathbb{A}^m$ be a map such that each coordinate function is a polynomial. Then ϕ is regular.

Definition 2.5.4. Let $\phi : V \to W$ be a bijective regular map of of (quasi-) affine varieties, and suppose that the inverse $\phi^{-1} : W \to V$ is also regular. Then, ϕ is said to be an isomorphism of varieties, and the varieties V and W are said to be isomorphic.

The following exercise demonstrates a point of caution.

Problem 2.5.3. Let $V \subset \mathbb{A}^2$ be the variety defined by the equation $x_2^2 = x_1^3$. Then, there is a natural regular map $\phi : \mathbb{A}^1 \to V$ that sends $t \to (t^2, t^3)$. Show that ϕ is bijective and both ϕ and ϕ^{-1} are continuous, but is not an isomorphism. (Hint: You may use the fact that every closed subset of V is in fact a finite collection of points)

We now turn to the projective case; considerable care must be taken to get a meaningful definition of "regular map".

Definition 2.5.5. Let $X \subset \mathbb{P}^n$ be a projective variety and let U be a Zariski-open subset of X. Recall the "standard affine chart" of $\mathbb{P}^n$ denoted by the collection $\{U_i\}$. Identifying each U_i with $\mathbb{A}^n$, we say a function $U \to \mathbb{A}^1$ is regular if it is locally regular; i.e. f restricted to $U_i \cap U$ is a regular function (in the affine sense) for each i.

Definition 2.5.6. Let $V \subset \mathbb{P}^n$ and $W \subset \mathbb{P}^m$ be (quasi-) projective varieties, and let $\phi : V \to W$ be a map. Furthermore, let $\{U_i\}$ be the standard affine chart of $\mathbb{P}^m$, and let $\{U_j'\}$ be the standard affine chart of $\mathbb{P}^n$. ϕ is said to be regular if for each affine $U_i \subset \mathbb{P}^m$, the restriction of ϕ to $\phi^{-1}(U_i) \to U_i$ is locally regular; i.e. on each affine $U_j' \subset \mathbb{P}^n$, the map $U_j' \cap \phi^{-1}(U_i) \to U_i$ is a regular map in the affine sense.

The above definition is extremely convoluted and difficult to work with; we give an easier criterion for determining regularity below.

Lemma 2.5.7. *Let $V \subset \mathbb{P}^n$ and $W \subset \mathbb{P}^m$ be (quasi-) projective varieties. Let $\phi : V \to W$ be a map that is given by $[X_0, \cdots, X_n] \to [P_0(X_0, \cdots, X_n), \cdots, P_m(X_0, \cdots, X_n)]$ where $P_0, \cdots, P_m$ are homogeneous polynomials of common degree d that do not simultaneously vanish at any point in $\mathbb{P}^n$. Then ϕ is a regular map.*

Not all regular maps may be characterized in the above manner; in some cases, even when the polynomials $P_0, \cdots, P_m$ have a common root, the map ϕ can be "extended" to a regular map. We state one striking fact about regular maps of projective varieties:

Theorem 2.5.8. *Let $\phi : V \to \mathbb{P}^n$ be a regular map. Then $\phi(V) \subset \mathbb{P}^n$ is a projective variety.*

The analogous statement in the affine case is not true; for instance, consider the map $f : \mathbb{A}^2 \to \mathbb{A}^2$ given by $(x, y) \to (x, xy)$. Then f is clearly regular; but the image of f is not an affine variety.

Problem 2.5.4. Prove rigorously that the image of the map f is not an affine variety. (Hint: Is the image of f classically closed?)

One beautiful consequence of Theorem 2.5.8:

Problem 2.5.5. Let V be a projective variety, and let $f : V \to \mathbb{A}^1$ be a regular function on V. Show that $f(V)$ is a finite collection of points. (Hint: Choose an embedding $\phi : \mathbb{A}^1 \to \mathbb{P}^1$, i.e. identify $\mathbb{A}^1$ with some standard affine subset of $\mathbb{P}^1$. Then consider the composition $\phi \circ f : V \to \mathbb{P}^1$, and argue that $\phi \circ f$ is regular. Now apply Theorem 2.5.8 to conclude that $\phi \circ f(V)$ is a projective variety. What are the projective subvarieties of $\mathbb{P}^1$?)

2.6 27 lines Theorem on the Fermat Surface

The goal of this section is for you to prove the following:

Theorem 2.6.1. *Let $V \subset \mathbb{P}^3$ be the variety defined by the equation $X_0^3 + X_1^3 + X_2^3 + X_3^3 = 0$. Then V contains exactly 27 lines.*

To do this, we will break up the proof into steps. First, show the following:

Problem 2.6.1. Let l be a line in $\mathbb{P}^3$. Show that l may be described as the common zero locus of 2 linear polynomials. (Hint: Linear Algebra.)

Now, show the following:

Problem 2.6.2. Let l be a line in $\mathbb{P}^3$. Show that, up to a permutation of coordinates, l may be defined by the equations

$$X_0 = \alpha_2 X_2 + \alpha_3 X_3$$
$$X_1 = \beta_2 X_2 + \beta_3 X_3$$

Problem 2.6.3. Let l be a line in $\mathbb{P}^3$ of the form

$$X_0 = \alpha_2 X_2 + \alpha_3 X_3$$
$$X_1 = \beta_2 X_2 + \beta_3 X_3$$

Show that l is contained in the Fermat surface if and only if

$$\alpha_2^3 + \beta_2^3 + 1 = 0$$
$$\alpha_3^3 + \beta_3^3 + 1 = 0$$
$$\alpha_2^2\alpha_3 + \beta_2^2\beta_3 = 0$$
$$\alpha_2\alpha_3^2 + \beta_2\beta_3^2 = 0$$

Problem 2.6.4. Consider the system of equations

$$\alpha_2^3 + \beta_2^3 + 1 = 0$$
$$\alpha_3^3 + \beta_3^3 + 1 = 0$$
$$\alpha_2^2\alpha_3 + \beta_2^2\beta_3 = 0$$
$$\alpha_2\alpha_3^2 + \beta_2\beta_3^2 = 0$$

where $\alpha_2, \alpha_3, \beta_2, \beta_3 \in \mathbb{C}$. Show that this system of equations has no solutions if $\alpha_2, \alpha_3, \beta_2, \beta_3$ are all nonzero. Furthermore, show that this system of equations has exactly 18 solutions.

Notice that the equation $X_0^3 + X_1^3 + X_2^3 + X_3^3 = 0$ is invariant under permutation of coordinates. With this in mind, prove the following:

Problem 2.6.5. Show that the Fermat surface contains exactly 27 lines. (Possible Hint: Produce $18 \times 6 = 108$ pairs of equations such that the lines defined by these pairs of equations all lie in the Fermat surface. Let $\mathcal{L}$ be the set of all such pairs of equations. Now, if l is a line in the Fermat surface, there will exactly 4 pairs of equations in $\mathcal{L}$ that define l. Therefore the number of lines in the Fermat surface is $\frac{18\times 6}{4} = 27$.)

As a final note, we state (without proof) the 27 lines Theorem in full generality:

Theorem 2.6.2. *(27 lines on a cubic surface) Let $V \subset \mathbb{P}^3$ be a smooth*[3] *cubic (hyper)surface in $\mathbb{P}^3$. Then V contains exactly 27 lines.*

This concludes the power round. Congratulations on finishing!

[3]We haven't defined what it means for a variety to be smooth. Intuitively, you can think of a smooth variety as having a "well-behaved" tangent space at each point.

Team Number:

PUMaC 2023 Power Round Cover Sheet

Remember that this sheet comes first in your stapled solutions. You should submit solutions for the problems in increasing order. Write on one side of the page only. The start of a solution to a problem should start on a new page. Please mark which questions for which you submitted a solution to help us keep track of your solutions.

Problem Number	Points	Attempted?
1.3.1	10	
1.3.2	5	
1.3.3	5	
1.3.4	5	
1.3.5	10	
1.3.6	15	
1.3.7	5	
1.3.8	5	
1.3.9	10	
1.3.10	10	
1.3.11	10	
1.3.12	25	
1.3.13	20	
1.4.1	15	
1.4.2	15	
1.4.3	15	
1.4.4	5	
1.4.5	5	
1.4.6	15	
1.4.7	10	
1.4.8	25	
1.4.9	10	
2.1.1	5	
2.2.1	10	
2.2.2	15	
2.2.3	15	
2.2.4	10	
2.2.5	5	
2.3.1	20	
2.3.2	10	
2.3.3	25	
2.3.4	15	
2.3.5	10	

Problem Number	Points	Attempted?
2.3.6	15	
2.3.7	15	
2.3.8	5	
2.4.1	10	
2.4.2	20	
2.4.3	30	
2.4.4	5	
2.4.5	15	
2.4.6	10	
2.4.7	10	
2.5.1	5	
2.5.2	35	
2.5.3	25	
2.5.4	15	
2.5.5	25	
2.6.1	10	
2.6.2	20	
2.6.3	20	
2.6.4	30	
2.6.5	40	