Number Theory B Solutions

1. Suppose that the greatest common divisor of n and 5040 is equal to 120. Determine the sum of the four smallest possible positive integers n.

   Proposed by Frank Lu

   Answer: $\boxed{3600}$

   Note that for the greatest common divisor of n and 5040 to equal 120, we need $n = 120d$, where d is relatively prime to $\frac{5040}{120} = 42$. But then note that this means that d can't be divisible by $2, 3$, or 7. This yields us that $d = 1, 5, 11, 13$, yielding the sum of n as $120(1 + 5 + 11 + 13) = 120(30) = 3600$.

2. Find the sum of the 23 smallest positive integers that are 4 more than a multiple of 23 and whose last two digits are 23.

   Proposed by Julian Shah

   Answer: $\boxed{610029}$

   We want the 23 smallest integers congruent to $23 \pmod{100}$ and $4 \pmod{23}$. We can use CRT to find that such integers must be $1223 \pmod{2300}$. Our answer is

$$23 \cdot 1223 + 2300(0 + 1 + \ldots + 22) = 23 \cdot 1223 + \frac{1}{2} \cdot 2300 \cdot 22 \cdot 23 = 610029$$

3. Find the sum of all prime numbers p such that p divides

$$(p^2 + p + 20)^{p^2+p+2} + 4(p^2 + p + 22)^{p^2-p+4}.$$

   Proposed by Sunay Joshi

   Answer: $\boxed{344}$

   We claim that the primes are $p = 2, 61, 281$, yielding an answer of $2 + 61 + 281 = 344$. First, the expression is congruent to $20^4 + 4 \cdot 22^4$ modulo p by Fermat's Little Theorem. Next, note that by the Sophie-Germain Identity, we can rewrite the expression as $2^4 \cdot (10^4 + 4 \cdot 11^4) = 2^4 \cdot (10^2 + 2 \cdot 11^2 - 2 \cdot 10 \cdot 11)(10^2 + 2 \cdot 11^2 + 2 \cdot 10 \cdot 11)$, which equals $2^6 \cdot 61 \cdot 281$. Since p divides this product, p must be among $\{2, 61, 281\}$, and the result follows.

4. Compute the sum of all positive integers whose positive divisors sum to 186.

   Proposed by Nancy Xu

   Answer: $\boxed{202}$

   The sum of the divisors of an integer with prime factorization $p_1^{n_1} p_2^{n_2} \ldots p_k^{n_k}$ is given by $(1 + p_1 + \ldots p_1^{n_1})(1 + p_2 + \ldots p_2^{n_2}) \ldots (1 + p_k + \ldots p_k^{n_k})$. We see that $186 = 2 \cdot 3 \cdot 31$, so it has factors $1, 2, 3, 6, 31, 62, 93, 186$. It is clear that 1 and 2 cannot be written as the sum of powers of a prime, so by trying out small primes, the only remaining possibilities are $186 = 6 \cdot 31 = (1 + 5)(1 + 2 + 4 + 8 + 16)$ and $186 = 3 \cdot 62 = (1 + 2)(1 + 61)$. Thus our two numbers are $5 \cdot 16 = 80$ and $2 \cdot 61 = 122$, sum the sum is $80 + 122 = 202$.

5. Given $k \geq 1$, let p_k denote the k-th smallest prime number. If N is the number of ordered 4-tuples (a, b, c, d) of positive integers satisfying $abcd = \prod_{k=1}^{2023} p_k$ with $a < b$ and $c < d$, find $N \pmod{1000}$.

   Proposed by Sunay Joshi

Answer: $\boxed{112}$

We claim that if $n \geq 2$ is square-free, then the number of ordered 4-tuples (a, b, c, d) satisfying $abcd = n$ with $a < b$ and $c < d$ is exactly $\frac{1}{4}\tau(n)^2 - \frac{1}{2}\tau(n)$. To see this, note that a 4-tuple (a, b, c, d) coresponds to a choice of divisor $d_1 = ab$ of n. By symmetry, there are $\frac{\tau(d_1)}{2}$ ways to pick the pair (a, b) with $a < b$. Similarly there are $\frac{\tau(n/d_1)}{2}$ ways to pick (c, d) with $c < d$. Therefore the total number of 4-tuples is $(\sum_{d_1|n} \frac{\tau(d_1)}{2}\frac{\tau(n/d_1)}{2}) - 2 \cdot \frac{\tau(1)}{2} \cdot \frac{\tau(n)}{2}$, where we subtract the terms corresponding to $d_1 = 1, n$. Since n is square-free, we have $\gcd(d_1, n/d_1) = 1$, hence $\tau(d_1)\tau(n/d_1) = \tau(n)$ and the above reduces to $\frac{1}{4}\tau(n)^2 - \frac{1}{2}\tau(n)$, as claimed.

Returning to the problem, note that for $n = \prod_{k=1}^{2023} p_k$, we have $\tau(n) = 2^{2023}$, hence $N = 2^{2\cdot 2023-2} - 2^{2023-1} = 2^{2022}(2^{2022} - 1)$. This is clearly $0 \pmod 8$. By Euler's Theorem, we see that $N \equiv 2^{22}(2^{22} - 1) \equiv 48^2(48^2 - 1) \equiv 112 \pmod{125}$. By the Chinese Remainder Theorem, $N \equiv 112 \pmod{1000}$, our answer.

6. Find the number of ordered pairs (x, y) of integers with $0 \leq x < 2023$ and $0 \leq y < 2023$ such that $y^3 \equiv x^2 \pmod{2023}$.

   Proposed by Brandon Cho

   Answer: $\boxed{3927}$

   Since $2023 = 7 \cdot 17^2$, by the Chinese Remainder Theorem it suffices to consider the pair of congruences $y^3 \equiv x^2 \pmod 7$ and $y^3 \equiv x^2 \pmod{17^2}$.

   For the former, note that since $x^2 \in \{0, 1, 2, 4\}$ and $y^3 \in \{0, 1, -1\}$, we must have $y^3 \equiv x^2 \equiv 0$ or $y^3 \equiv x^2 \equiv 1$. The former corresponds to $(0, 0)$. The latter is satisfied when $x \in \{1, -1\}$ and $y \in \{1, 2, 4\}$. This yields 6 pairs. Thus this case has 7 solutions.

   For the latter congruence, we consider two cases. The first case is when 17 does not divide y, so that 17 does not divide x. Further the map $y \mapsto y^3$ is a bijection of the set of units of $\mathbb{Z}/17^2\mathbb{Z}$. Therefore each choice of unit x corresponds to a unique solution for y. Since there are $17^2 - 17$ units mod 17^2, we have a total of $17^2 - 17$ pairs in this case. The second case is when 17 divides y, hence 17 divides x. Any such pair (x, y) satisfies the congruence since both sides are 0. It follows that there are $17 \cdot 17$ pairs in this third case. Summing, we find $2 \cdot 17^2 - 17$ pairs.

   Finally, we multiply the number of solutions to each of the two congruences to find an answer of $7 \cdot (2 \cdot 17^2 - 17) = 3927$.

7. A positive integer $\ell \geq 2$ is called *sweet* if there exists a positive integer $n \geq 10$ such that when the leftmost nonzero decimal digit of n is deleted, the resulting number m satisfies $n = m\ell$. Let S denote the set of all sweet numbers ℓ. If the sum $\sum_{\ell \in S} \frac{1}{\ell - 1}$ can be written as $\frac{A}{B}$ for relatively prime positive integers A, B, find $A + B$.

   Proposed by Sunay Joshi

   Answer: $\boxed{71}$

   Let $\nu_p(t)$ denote the highest power of the prime p dividing t. We claim that $\ell \geq 2$ is sweet iff: (i) all prime factors of $\ell - 1$ are elements of $\{2, 3, 5, 7\}$, (ii) $\nu_3(\ell - 1) \leq 2$, (iii) $\nu_7(\ell - 1) \leq 1$, (iv) $3 \cdot 7$ does not divide $\ell - 1$, and (v) $\ell - 1 \neq 1, 3, 7, 9$. To see this, suppose that $n = m\ell$, where m is the number obtained by deleting the leftmost digit of n. Write $n = 10^k a + b$, where $a \in \{1, \ldots, 9\}$ is the leftmost digit of n, so that $m = b$. Then $n = m\ell$ is equivalent to $10^k a + b = \ell b$, or $(\ell - 1)b = 10^k a$ for some k-digit number b.

   The condition $(\ell - 1)b = 10^k a$ for an arbitrary positive integer b is equivalent to $(\ell - 1)|10^k a$ for some $a \in \{1, \ldots, 9\}$, which is equivalent to the first four conditions above.

If $\ell - 1 \geq 10$, then $(\ell-1)b = 10^k a$ for some k-digit number b is equivalent to $(\ell-1)|10^k a$, since the equality forces b to have at most k digits: $b \leq 10^k \cdot 9/10 < 10^k$. If $\ell - 1 \in \{1, \dots, 9\}$, then in the cases $\ell - 1 \in \{1, 3, 7, 9\}$, b must have at least k digits. The value of b in each case is at least $\frac{10^k}{1}$, $\frac{3\cdot 10^k}{3}$, $\frac{7\cdot 10^k}{7}$, and $\frac{9\cdot 10^k}{9}$, respectively.

Thus $\ell \in S$ iff the five conditions above hold. In terms of prime factorization, $\ell \in S$ iff $\ell - 1 \neq 1, 3, 7, 9$ and $\ell - 1 = 2^x 5^y 3^z 7^w$, where $x \geq 0$, $y \geq 0$, and $(z, w) \in \{(0,0), (0,1), (1,0), (2,0)\}$. Splitting the desired sum into a product over primes, we find

$$\sum_{\ell \in S} \frac{1}{\ell - 1} = \left(\sum_{x \geq 0} \frac{1}{2^x}\right)\left(\sum_{y \geq 0} \frac{1}{5^y}\right)\left(1 + \frac{1}{3} + \frac{1}{3^2} + \frac{1}{7}\right) - \left(\frac{1}{1} + \frac{1}{3} + \frac{1}{7} + \frac{1}{9}\right),$$

where we subtract terms corresponding to the cases $\ell - 1 = 1, 3, 7, 9$. By the geometric series formula, this equals $\frac{250}{63} - \frac{100}{63} = \frac{50}{21}$. Thus our answer is $50 + 21 = 71$.

8. Given a positive integer ℓ, define the sequence $\{a_n^{(\ell)}\}_{n=1}^{\infty}$ such that $a_n^{(\ell)} = \lfloor n + \sqrt[\ell]{n} + \frac{1}{2} \rfloor$ for all positive integers n. Let S denote the set of positive integers that appear in all three of the sequences $\{a_n^{(2)}\}_{n=1}^{\infty}$, $\{a_n^{(3)}\}_{n=1}^{\infty}$, and $\{a_n^{(4)}\}_{n=1}^{\infty}$. Find the sum of the elements of S that lie in the interval $[1, 100]$.

Proposed by Sunay Joshi

Answer: $\boxed{4451}$

We claim that a number $k+1$ is skipped by the sequence $\{a_n^{(\ell)}\}_{n=1}^{\infty}$ iff $k + 1 = m + \lceil (m + \frac{1}{2})^\ell \rceil$ for some $m \geq 0$. To see this, suppose $k+1$ is skipped by the sequence, so that $a_n = k$ and $a_{n+1} \geq k+2$. The condition $a_n = k$ is equivalent to $k \leq n + \sqrt[\ell]{n} + \frac{1}{2} < k + 1$ and thus $(m - \frac{1}{2})^k \leq n < (m + \frac{1}{2})^\ell$, where $m = k - n$. The condition $a_{n+1} \geq k + 2$ is equivalent to $k + 2 \leq (n+1) + \sqrt[\ell]{n+1} + \frac{1}{2}$, which can be rewritten as $(m + \frac{1}{2})^\ell - 1 \leq n$. Combining these two inequality chains, we find that $n = \lceil (m + \frac{1}{2})^\ell \rceil - 1$, hence the skipped number is $k + 1 = m + \lceil (m + \frac{1}{2})^\ell \rceil$, as claimed.

It follows that the numbers skipped in the sequence for $\ell = 2$ are $m + \lceil m^2 + m + \frac{1}{4} \rceil = (m+1)^2$; the numbers skipped for $\ell = 3$ are $m + \lceil m^3 + \frac{3}{2}m^2 + \frac{3}{4}m + \frac{1}{8} \rceil = m + m^3 + \lceil \frac{3}{2}m^2 + \frac{3}{4}m + \frac{1}{8} \rceil$; and the numbers skipped for $\ell = 4$ are $m + \lceil m^4 + 2m^3 + \frac{3}{2}m^2 + \frac{1}{2}m + \frac{1}{16} \rceil = m + m^4 + 2m^3 + \lceil \frac{3}{2}m^2 + \frac{1}{2}m + \frac{1}{16} \rceil$. The skipped numbers for $\ell = 2$ are $1, 4, 9, 16, 25, 36, 49, 64, 81, 100$, the skipped numbers for $\ell = 3$ are $1, 5, 18, 46, 96$, and the skipped numbers for $\ell = 4$ are $1, 7, 42$. The sum of (distinct) numbers that are skipped in at least one of the sequences can be seen to be 599, hence the sum of the numbers in $[1, 100]$ that are not skipped in any list is $5050 - 599 = 4451$, our answer.