Geometry B Solutions

1. Rectangle $ABCD$ has $AB = 24$ and $BC = 7$. Let d be the distance between the centers of the incircles of $\triangle ABC$ and $\triangle CDA$. Find d^2.

   Proposed by Atharva Pathak

   Answer: $\boxed{325}$

   Since $\triangle ABC$ has sidelengths $7, 24, 25$, the identity $K = rs$ implies $r = \frac{\frac{1}{2}\cdot 7\cdot 24}{\frac{7+24+25}{2}} = 3$. Since $\triangle ABC$ and $\triangle CDA$ are congruent, the inradius of $\triangle CDA$ is also 3. Thus the horizontal distance between the incenters is $24 - 3 - 3 = 18$, and the vertical distance between the incenters is $7 - 3 - 3 = 1$. By Pythagoras, our answer is $d^2 = 18^2 + 1^2 = 325$.

2. The area of the largest square that can be inscribed in a regular hexagon with sidelength 1 can be expressed as $a - b\sqrt{c}$ where c is not divisible by the square of any prime. Find $a+b+c$.

   Proposed by Adam Huang

   Answer: $\boxed{21}$

   Let our regular hexagon be $ABCDEF$ with center O. It is easy to see that the largest square must be congruent to a square $WXYZ$ centered at O, where W, X, Y, Z lie on sides AB, CD, DE, FA such that $WX \parallel FB$ and $XY \parallel BC$. Let $c = AW$, $b = WB$, and $d = WX$. Clearly $b + c = 1$. By drawing an altitude from A in $\triangle ZAW$, we find $d = c\sqrt{3}$. By drawing altitudes from B, C in trapezoid $BCXW$, we find $d = \frac{b}{2} + 1 + \frac{b}{2} = b+1$. Therefore $c\sqrt{3} = b+1$, so that $c(\sqrt{3}+1) = 2$, and so $c = \sqrt{3} - 1$. Hence $d = 3 - \sqrt{3}$, which yields an area of $(3-\sqrt{3})^2 = 12 - 6\sqrt{3}$ and an answer of $12 + 6 + 3 = 21$.

3. Define a *common chord* between two intersecting circles to be the line segment connecting their two intersection points. Let $\omega_1, \omega_2, \omega_3$ be three circles of radii 3, 5, and 7, respectively. Suppose they are arranged in such a way that the common chord of ω_1 and ω_2 is a diameter of ω_1, the common chord of ω_1 and ω_3 is a diameter of ω_1, and the common chord of ω_2 and ω_3 is a diameter of ω_2. Compute the square of the area of the triangle formed by the centers of the three circles.

   Proposed by Eric Shen

   Answer: $\boxed{96}$

   By Pythagoras, the distance between the centers of circles ω_i and ω_j with $j > i$ is $\sqrt{r_j^2 - r_i^2}$. We seek the area of a triangle with sidelengths $\sqrt{16}$, $\sqrt{24}$, and $\sqrt{40}$. But this is a right triangle whose area is $\frac{1}{2}\cdot\sqrt{16}\cdot\sqrt{24} = 4\sqrt{6}$, and our answer is $(4\sqrt{6})^2 = 96$.

4. Let $\triangle ABC$ be an isosceles triangle with $AB = AC = \sqrt{7}$ and $BC = 1$. Let G be the centroid of $\triangle ABC$. Given $j \in \{0, 1, 2\}$, let T_j denote the triangle obtained by rotating $\triangle ABC$ about G by $2\pi j/3$ radians. Let $\mathcal{P}$ denote the intersection of the interiors of triangles T_0, T_1, T_2. If K denotes the area of $\mathcal{P}$, then $K^2 = \frac{a}{b}$ for relatively prime positive integers a, b. Find $a + b$.

   Proposed by Sunay Joshi

   Answer: $\boxed{1843}$

   Construct the equilateral triangle $\triangle AXY$ with sidelength 3 such that BC is the middle third of the side XY (with B closer to X, WLOG). Note that T_0, T_1, T_2 lie within $\triangle AXY$; they are simply $\triangle ABC$ rotated. Let M, N lie on AY, AX such that $AM/MY = 1/2$ and $AN/NX = 1/2$. Let $P = AB \cap MX$, $Q = NY \cap MX$, and $R = AC \cap NY$. Then it is easy to see by symmetry that we see $K = 3 \cdot [PQRG]$. Since $PQRG$ has perpendicular diagonals, its area is given by $\frac{1}{2} \cdot QG \cdot PR$. To compute PR, note by mass points that $AP/PB = 3/2$, hence by similar triangles $PR = 3/5 \cdot BC = 3/5$. By mass points, we also have hat AQ is half the height of $\triangle AXY$, hence $QG = AG - AQ = \sqrt{3}/4$. Solving for K, we find $K = 3/2 \cdot \sqrt{3}/4 \cdot 3/5 = 9\sqrt{3}/40$. Squaring yields $K^2 = 243/1600$ and our answer is $243 + 1600 = 1843$.

5. Let $\triangle ABC$ be a triangle with $AB = 13$, $BC = 14$, and $CA = 15$. Let D, E, and F be the midpoints of AB, BC, and CA respectively. Imagine cutting $\triangle ABC$ out of paper and then folding $\triangle AFD$ up along FD, folding $\triangle BED$ up along DE, and folding $\triangle CEF$ up along EF until A, B, and C coincide at a point G. The volume of the tetrahedron formed by vertices D, E, F, and G can be expressed as $\frac{p\sqrt{q}}{r}$, where p, q, and r are positive integers, p and r are relatively prime, and q is square-free. Find $p + q + r$.

   Proposed by Atharva Pathak

   Answer: $\boxed{80}$

   Let H_1 be the foot of the perpendicular from A to DF and let H_2 be the foot of the perpendicular from E to DF. Note that a 13-14-15 triangle is a 5-12-13 triangle glued to a 9-12-15 triangle along the side of length 12. Because $\triangle ADF$ and $\triangle EFD$ are similar to $\triangle ABC$ scaled by a factor of $1/2$, we get that $AH_1 = EH_2 = 6$, $DH_1 = FH_2 = \frac{5}{2}$, and $H_1H_2 = 2$. Let θ be the dihedral angle between $\triangle GDF$ and $\triangle EDF$ in the tetrahedron. Because GE came from BE and CE in the original triangle, we have $GE = 7$. Now imagine projecting points G, H_1, H_2, and E onto a plane perpendicular to FD, such that G maps to G', H_1 and H_2 map to H', and E maps to E'. Since GE has a component of length $H_1H_2 = 2$ perpendicular to the plane, we get $G'E' = \sqrt{7^2 - 2^2} = \sqrt{45}$. Applying the law of cosines to $\triangle G'H'E'$ with the angle θ at H' gives $\cos\theta = \frac{3}{8}$. So the height of the tetrahedron, which is the distance from G' to $H'E'$, is $6\sin\theta = \frac{3\sqrt{55}}{4}$. Finally, the area of the base of the tetrahedron, i.e. $\triangle DEF$, is $\frac{1}{2}(7)(6) = 21$, so the volume is $\frac{1}{3}(21)\left(\frac{3\sqrt{55}}{4}\right) = \frac{21\sqrt{55}}{4}$, which gives a final answer of 80.

6. Let $\triangle ABC$ be a triangle with $AB = 4$, $BC = 6$, and $CA = 5$. Let the angle bisector of $\angle BAC$ intersect BC at the point D and the circumcircle of $\triangle ABC$ again at the point $M \neq A$. The perpendicular bisector of segment DM intersects the circle centered at M passing through B at two points, X and Y. Compute $AX \cdot AY$.

   Proposed by Eric Shen

   Answer: $\boxed{36}$

   Note that $AX = AY$ by symmetry and that $AX = AM$ by inversion about M. In a 4-5-6 triangle we have the following relation between the angles: $A = 2C$. Since AM subtends an angle of $\frac{A}{2} + C$ and since $\frac{A}{2} + C = A$, it follows that $AM = BC = 6$. Our answer is $6^2 = 36$.

7. Let $\triangle ABC$ have $AB = 15$, $AC = 20$, and $BC = 21$. Suppose ω is a circle passing through A that is tangent to segment BC. Let point $D \neq A$ be the second intersection of AB with ω, and let point $E \neq A$ be the second intersection of AC with ω. Suppose DE is parallel to BC. If $DE = \frac{a}{b}$, where a, b are relatively prime positive integers, find $a + b$.

   Proposed by Frank Lu

   Answer: $\boxed{361}$

   First, since DE is parallel to BC, we have that triangles ADE, ABC are similar. Furthermore, we have a homothety that sends triangle ADE to ABC. Notice that the image of this homothety also sends ω to the circumcircle of ABC. We thus need to determine the ratio of this homothety.

   To do this, let X be the tangency point of ω to BC. Draw line AX, and let M be the second intersection of line AX with the circumcircle. Then, we know from homothety that $\frac{AX}{AM} = \frac{DE}{BC}$;

we just need to compute AX, AM. We furthermore note that ω is tangent to the circumcircle by homothety. From this configuration, we thus find that M is the midpoint of the minor arc BC, meaning that AX is the angle bisector of angle $\angle BAC$. First, to compute AX, we can employ Stewart's theorem. We know from the angle bisector theorem that $\frac{BX}{CX} = \frac{3}{4}$, meaning that $BX = 9$ and $CX = 12$. Therefore, we have by Stewart's theorem that $AX^2 \cdot 21 + 9 \cdot 12 \cdot 21 = 15^2 \cdot 12 + 20^2 \cdot 9$, or that $21AX^2 = 5^2 \cdot 3 \cdot (3^2 \cdot 4 + 4^2 \cdot 3) - 9 \cdot 12 \cdot 21 = 5^2 \cdot 3 \cdot 4 \cdot 21 - 9 \cdot 12 \cdot 21$, or that $AX^2 = 300 - 108 = 192$, so $AX = 8\sqrt{3}$.

From here, we compute AM. The method that we use to compute this is Ptolemy's theorem and Law of Cosines chasing. First, consider BM, CM. Note that $\angle BMC = 180 - \angle BAC$, and so $\cos\angle BAC = -\cos\angle BMC$. But now by Law of Cosines, we know that $\cos\angle BAC = \frac{15^2+20^2-21^2}{2\cdot 15\cdot 20} = \frac{23}{75}$. Therefore, we have that $BC^2 = BM^2(2-2\cos\angle BMC) = BM^2\frac{196}{75}$, meaning that $BM = CM = \frac{15\sqrt{3}}{2}$. Finally, by Ptolemy's Theorem, we have that $AM \cdot BC = BM(AB + AC)$, or that $AM = \frac{35}{21}\frac{15\sqrt{3}}{2} = \frac{25\sqrt{3}}{2}$.

It follows that $DE = \frac{AX}{AM}BC = \frac{336}{25}$, so our answer is 361.

8. Let $\triangle ABC$ have $AB = 14$, $BC = 30$, $AC = 40$ and $\triangle AB'C'$ with $AB' = 7\sqrt{6}$, $B'C' = 15\sqrt{6}$, $AC' = 20\sqrt{6}$ such that $\angle BAB' = \frac{5\pi}{12}$. The lines BB' and CC' intersect at point D. Let O be the circumcenter of $\triangle BCD$, and let O' be the circumcenter of $\triangle B'C'D$. Then the length of segment OO' can be expressed as $\frac{a+b\sqrt{c}}{d}$, where a, b, c, and d are positive integers such that a and d are relatively prime, and c is not divisible by the square of any prime. Find $a+b+c+d$.

Proposed by Adam Huang

Answer: $\boxed{55}$

Note that $\triangle ABC$ and $\triangle AB'C'$ are spirally similar with center of spiral similarity given by A and angle $\frac{5\pi}{12}$ and dilation factor $\frac{\sqrt{6}}{2}$. By properties of spiral similarity, we have that $D := BB' \cap CC'$ lies on circumcircles (ABC) and $(AB'C')$. Therefore AO is the circumradius of $\triangle ABC$, and $AO' = \frac{\sqrt{6}}{2}AO$ by similarity, with $\angle OAO' = \frac{5\pi}{12}$. To compute $R := AO$, note by Heron that the area of $\triangle ABC$ is $K = 168$, so that $\frac{abc}{4R} = K \implies R = \frac{abc}{4K} = \frac{14\cdot 30\cdot 40}{4\cdot 168} = 25$. By Law of Cosines, we have $(OO')^2 = 25^2 \cdot (1^2 + (\frac{\sqrt{6}}{2})^2 - 2\cdot 1\cdot \frac{\sqrt{6}}{2}\cdot \cos\frac{5\pi}{12})$, so that $OO' = \frac{25+25\sqrt{3}}{2}$, which yields an answer of $a+b+c+d = 25+25+3+2 = 55$.