Geometry A

1. Define a *common chord* between two intersecting circles to be the line segment connecting their two intersection points. Let $\omega_1, \omega_2, \omega_3$ be three circles of radii 3, 5, and 7, respectively. Suppose they are arranged in such a way that the common chord of ω_1 and ω_2 is a diameter of ω_1, the common chord of ω_1 and ω_3 is a diameter of ω_1, and the common chord of ω_2 and ω_3 is a diameter of ω_2. Compute the square of the area of the triangle formed by the centers of the three circles.

2. Let $\triangle ABC$ be an isosceles triangle with $AB = AC = \sqrt{7}$ and $BC = 1$. Let G be the centroid of $\triangle ABC$. Given $j \in \{0, 1, 2\}$, let T_j denote the triangle obtained by rotating $\triangle ABC$ about G by $2\pi j/3$ radians. Let $\mathcal{P}$ denote the intersection of the interiors of triangles T_0, T_1, T_2. If K denotes the area of $\mathcal{P}$, then $K^2 = \frac{a}{b}$ for relatively prime positive integers a, b. Find $a + b$.

3. Let $\triangle ABC$ be a triangle with $AB = 13$, $BC = 14$, and $CA = 15$. Let D, E, and F be the midpoints of AB, BC, and CA respectively. Imagine cutting $\triangle ABC$ out of paper and then folding $\triangle AFD$ up along FD, folding $\triangle BED$ up along DE, and folding $\triangle CEF$ up along EF until A, B, and C coincide at a point G. The volume of the tetrahedron formed by vertices D, E, F, and G can be expressed as $\frac{p\sqrt{q}}{r}$, where p, q, and r are positive integers, p and r are relatively prime, and q is square-free. Find $p + q + r$.

4. Let $\triangle ABC$ be a triangle with $AB = 4$, $BC = 6$, and $CA = 5$. Let the angle bisector of $\angle BAC$ intersect BC at the point D and the circumcircle of $\triangle ABC$ again at the point $M \neq A$. The perpendicular bisector of segment DM intersects the circle centered at M passing through B at two points, X and Y. Compute $AX \cdot AY$.

5. Let $\triangle ABC$ have $AB = 15$, $AC = 20$, and $BC = 21$. Suppose ω is a circle passing through A that is tangent to segment BC. Let point $D \neq A$ be the second intersection of AB with ω, and let point $E \neq A$ be the second intersection of AC with ω. Suppose DE is parallel to BC. If $DE = \frac{a}{b}$, where a, b are relatively prime positive integers, find $a + b$.

6. Let $\triangle ABC$ have $AB = 14$, $BC = 30$, $AC = 40$ and $\triangle AB'C'$ with $AB' = 7\sqrt{6}$, $B'C' = 15\sqrt{6}$, $AC' = 20\sqrt{6}$ such that $\angle BAB' = \frac{5\pi}{12}$. The lines BB' and CC' intersect at point D. Let O be the circumcenter of $\triangle BCD$, and let O' be the circumcenter of $\triangle B'C'D$. Then the length of segment OO' can be expressed as $\frac{a+b\sqrt{c}}{d}$, where a, b, c, and d are positive integers such that a and d are relatively prime, and c is not divisible by the square of any prime. Find $a+b+c+d$.

7. Let $\triangle ABC$ be a triangle with $\angle BAC = 90^\circ$, $\angle ABC = 60^\circ$, and $\angle BCA = 30^\circ$ and $BC = 4$. Let the incircle of $\triangle ABC$ meet sides BC, CA, AB at points A_0, B_0, C_0, respectively. Let $\omega_A, \omega_B, \omega_C$ denote the circumcircles of triangles $\triangle B_0IC_0, \triangle C_0IA_0, \triangle A_0IB_0$, respectively. We construct triangle T_A as follows: let A_0B_0 meet ω_B for the second time at $A_1 \neq A_0$, let A_0C_0 meet ω_C for the second time at $A_2 \neq A_0$, and let T_A denote the triangle $\triangle A_0A_1A_2$. Construct triangles T_B, T_C similarly. If the sum of the areas of triangles T_A, T_B, T_C equals $\sqrt{m} - n$ for positive integers m, n, find $m + n$.

8. Similar to the last 6 problems, let $\triangle ABC$ be a triangle with $AB = 4$ and $AC = \frac{7}{2}$. Let ω denote the A-excircle of $\triangle ABC$. Let ω touch lines AB, AC at the points D, E, respectively. Let Ω denote the circumcircle of $\triangle ADE$. Consider the line ℓ parallel to BC such that ℓ is tangent to ω at a point F and such that ℓ does not intersect Ω. Let ℓ intersect lines AB, AC at the points X, Y, respectively, with $XY = 18$ and $AX = 16$. Let the perpendicular bisector of XY meet the circumcircle of $\triangle AXY$ at P, Q, where the distance from P to F is smaller than the distance from Q to F. Let ray $\overrightarrow{PF}$ meet Ω for the first time at the point Z. If $PZ^2 = \frac{m}{n}$ for relatively prime positive integers m, n, find $m + n$.

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(Write answers on next page.)
Name:

Team:

Write answers in table below:

Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8