P U M ∴ C

Geometry A Solutions

1. Circle Γ is centered at $(0,0)$ in the plane with radius $2022\sqrt{3}$. Circle Ω is centered on the x-axis, passes through the point $A = (6066, 0)$, and intersects Γ orthogonally at the point $P = (x, y)$ with $y > 0$. If the length of the minor arc AP on Ω can be expressed as $\frac{m\pi}{n}$ for relatively prime positive integers m, n, find $m + n$.

   (Two circles are said to intersect *orthogonally* at a point P if the tangent lines at P form a right angle.)

   Proposed by Sunay Joshi

   Answer: $\boxed{1349}$

   Let $O = (0,0)$. Let $R = 2022\sqrt{3}$ denote the radius of Γ, so that $OA = R\sqrt{3}$. Let r denote the radius of Ω. Let Q denote the center of Ω. Since OPQ is a right triangle, $PQ = \sqrt{r^2 + R^2}$. Since $OA = OQ + QA$, we have

   $$\sqrt{R^2 + r^2} + r = R\sqrt{3}$$

   Solving, we find that $r = R/\sqrt{3} = 2022$. Therefore $\angle OQP = 60^\circ$ and the minor arc AP corresponds to an interior angle of 120°. It follows that the desired arclength is given as $\frac{1}{3} \cdot 2\pi r = \frac{4044\pi}{3} = \frac{1348\pi}{1}$, and our answer is $1348 + 1 = 1349$.

2. An ellipse has foci A and B and has the property that there is some point C on the ellipse such that the area of the circle passing through A, B, and, C is equal to the area of the ellipse. Let e be the largest possible eccentricity of the ellipse. One may write e^2 as $\frac{a+\sqrt{b}}{c}$, where a, b, and c are integers such that a and c are relatively prime, and b is not divisible by the square of any prime. Find $a^2 + b^2 + c^2$.

   Proposed by Daniel Carter

   Answer: $\boxed{30}$

   Consider the ellipse with largest possible eccentricity that has this property. The smallest possible area of the circle is when the center of the circle is the center of the ellipse. Let O be the center of the ellipse. Then $\pi(OA)^2 = \pi Rr$, where R, r are the semi-major and semi-minor axes. We have $OA/R = e$, so then $(OA)e = r$. Noting that $r^2 = R^2 - (OA)^2$, this means $e^2 = (1/e^2 - 1)$, or $e^2 = \frac{-1+\sqrt{5}}{2}$. So the answer is $(-1)^2 + 5^2 + 2^2 = 30$.

3. Daeun draws a unit circle centered at the origin and inscribes within it a regular hexagon $ABCDEF$. Then Dylan chooses a point P within the circle of radius 2 centered at the origin. Let M be the maximum possible value of $|PA| \cdot |PB| \cdot |PC| \cdot |PD| \cdot |PE| \cdot |PF|$, and let N be the number of possible points P for which this maximal value is obtained. Find $M + N^2$.

   Proposed by Dylan Epstein-Gross

   Answer: $\boxed{101}$

   Using roots of unity, the product of lengths is

   $$|z-1||z-a||z-a^2|\cdots|z-a^5| = |z^6 - 1|$$

   This is maximized when $z^6 = -64$, which has six solutions with $M = 65$. Thus the answer is $65 + 6^2 = 101$.

4. Let $\triangle ABC$ be an equilateral triangle. Points D, E, F are drawn on sides AB, BC, and CA respectively such that $[ADF] = [BED] + [CEF]$ and $\triangle ADF \sim \triangle BED \sim \triangle CEF$. The ratio

$\frac{[ABC]}{[DEF]}$ can be expressed as $\frac{a+b\sqrt{c}}{d}$, where a, b, c, and d are positive integers such that a and d are relatively prime, and c is not divisible by the square of any prime. Find $a+b+c+d$.

(Here $[\mathcal{P}]$ denotes the area of polygon $\mathcal{P}$.)

Proposed by Adam Huang

Answer: $\boxed{17}$

Assume WLOG that $\triangle ABC$ has sidelength 1. The similarity condition implies $DE^2 + EF^2 = DF^2$, hence $\angle DEF = 90$. Angle chasing also yields $\angle BED = 45$, so that $\triangle BED, \triangle CEF$ are 60-45-75 triangles and $\triangle DEF$ is a 30-60-90 right triangle. By the Law of Sines applied to $\triangle BED$ and $\triangle CEF$, the lengths $z = DE$ and $x = BE$ satisfy $\frac{z}{\sin 60} = \frac{x}{\sin 75}$ and $\frac{z\sqrt{3}}{\sin 60} = \frac{1-x}{\sin 75}$. Solving, we find $x = \frac{\sqrt{3}-1}{2}$ and $z = \frac{2\sqrt{6}-3\sqrt{2}}{2}$. Thus $[DEF] = \frac{z^2\sqrt{3}}{2}$ and the ratio is $[ABC]/[DEF] = \frac{\sqrt{3}/4}{z^2\sqrt{3}/2}$, which reduces to $\frac{7+4\sqrt{3}}{3}$. Our answer is $7+4+3+3 = 17$.

5. Let $\triangle ABC$ be a triangle with $AB = 5$, $BC = 8$, and, $CA = 7$. Let the center of the A-excircle be O, and let the A-excircle touch lines BC, CA, and, AB at points X, Y, and, Z, respectively. Let h_1, h_2, and, h_3 denote the distances from O to lines XY, YZ, and, ZX, respectively. If $h_1^2 + h_2^2 + h_3^2$ can be written as $\frac{m}{n}$ for relatively prime positive integers m, n, find $m+n$.

   Proposed by Sunay Joshi

   Answer: $\boxed{2189}$

   Let a, b, c denote the lengths of sides BC, CA, AB, and let r_A denote the radius of the A-excircle. We claim that $h_1 = \frac{r_A^2 \sin\frac{C}{2}}{s-b}$, $h_2 = \frac{r_A^2 \sin\frac{B}{2}}{s-c}$, and $h_3 = \frac{r_A^2 \cos\frac{A}{2}}{s}$. We begin with h_1. Computing the area of $\triangle OXY$ in two ways, we find $\frac{1}{2}h_1 \cdot XY = \frac{1}{2}r_A^2 \sin XOY$. Since $XY = 2(s-b)\cos\frac{C}{2}$ and $\angle XOY = C$, solving the equation for h_1 yields the desired formula. By symmetry, this implies the expression for h_3. For h_2, we compute the area of $\triangle YOZ$ in two ways to find $\frac{1}{2}h_2 \cdot YZ = \frac{1}{2}r_A^2 \sin YOZ$. Since $\angle YOZ = \pi - A$ and $YZ = 2s\sin\frac{A}{2}$, solving the equation for h_2 yields the desired formula.

   Having established the above, we now compute each of h_1, h_2, h_3. By the Law of Cosines, $\cos A = \frac{1}{7}$, $\cos B = \frac{1}{2}$, and $\cos C = \frac{11}{14}$. By the half-angle formulae, it follows that $\cos\frac{A}{2} = \frac{2}{\sqrt{7}}$, $\sin\frac{B}{2} = \frac{1}{2}$, and $\sin\frac{C}{2} = \frac{\sqrt{3}}{2\sqrt{7}}$. Next, since $r_A(s-a) = K$, Heron's formula implies that $r_A^2 = \frac{s(s-b)(s-c)}{s-a} = 75$. Putting everything together, we find that

   $$h_1^2 + h_2^2 + h_3^2 = 75^2 \cdot \left[\left(\frac{2/\sqrt{7}}{10}\right)^2 + \left(\frac{\sqrt{3}/(2\sqrt{7})}{3}\right)^2 + \left(\frac{1/2}{5}\right)^2 \right] = \frac{2175}{14}$$

   This gives an answer of $m+n = 2189$.

6. Triangle $\triangle ABC$ has sidelengths $AB = 10, AC = 14$, and, $BC = 16$. Circle ω_1 is tangent to rays $\overrightarrow{AB}, \overrightarrow{AC}$ and passes through B. Circle ω_2 is tangent to rays $\overrightarrow{AB}, \overrightarrow{AC}$ and passes through C. Let ω_1, ω_2 intersect at points X, Y. The square of the perimeter of triangle $\triangle AXY$ is equal to $\frac{a+b\sqrt{c}}{d}$, where a, b, c, and, d are positive integers such that a and d are relatively prime, and c is not divisible by the square of any prime. Find $a+b+c+d$.

   Proposed by Frank Lu

   Answer: $\boxed{6272}$

   Draw the angle bisector of BAC, which we denote as ℓ. Notice that if O_1 is the center of ω_1 and O_2 is the center of ω_2, then we have that O_1, O_2 lie on this angle bisector. It follows that this angle bisector must be the perpendicular bisector of XY, since XY is the radical axis of these two circles.

We first compute AP. To do this, consider the following operation: first, reflect the diagram about the angle bisector, then perform an inversion about A of radius $\sqrt{10 \cdot 14}$. (This latter inversion is referred to sometimes as a root bc inversion). Notice that this operation sends the circle ω_1 to ω_2, and sends X to Y. Furthermore, since ℓ is the perpendicular bisector of XY, we have that $AY = AX = \frac{140}{AX}$, meaning that $AX = \sqrt{140}$. To find XP, we need to now find AP. But this can be done by considering the triangle O_1XO_2.

We compute the side lengths of this triangle. First, we know that XO_1 is the radius of ω_1. We can then compute XO_1 by considering the incircle: if the incircle has radius r, and l is the length of the tangent from A to the incircle, we know that $\frac{l}{r} = \frac{AB}{BO_1} = \frac{AB}{XO_1}$. But if s is the semi-perimeter of ABC, we know that $l = s - BC = 4$, and r is equal to $\sqrt{\frac{(s-AB)(s-BC)(s-AC)}{s}} = \sqrt{\frac{10 \cdot 6 \cdot 4}{20}} = 2\sqrt{3}$. Therefore, we see that $XO_1 = 5\sqrt{3}$. Similarly, $XO_2 = 7\sqrt{3}$. Finally, we can then compute that $O_1O_2 = AO_2 - AO_1$, which using the Pythagorean theorem is $7\sqrt{7} - 5\sqrt{7} = 2\sqrt{7}$.

Therefore, we compute that $PO_2^2 - PO_1^2 = 72$. Since this is larger than $2\sqrt{7}$, we see that our triangle is obtuse. Thus, we have that $O_1P + O_2P = \frac{36\sqrt{7}}{7}$, and $OP_2 - OP_1 = 2\sqrt{7}$, which gives us that $O_1P = \frac{11\sqrt{7}}{7}$, and so therefore $AP = AO_1 + O_1P = \frac{24\sqrt{7}}{7}$.

From here, we compute that $XP^2 = 140 - \frac{576}{7} = \frac{404}{7}$, or that $XP = 2\sqrt{\frac{101}{7}}$. Therefore, the perimeter of our triangle is equal to $4\sqrt{\frac{101}{7}} + 4\sqrt{35} = \frac{\sqrt{1616}+\sqrt{3920}}{\sqrt{7}}$. Therefore the square of the perimeter is $\frac{5536+224\sqrt{505}}{7}$, so our answer is $a + b + c + d = 5536 + 224 + 505 + 7 = 6272$.

7. Let $\triangle ABC$ be a triangle with $BC = 7$, $CA = 6$, and, $AB = 5$. Let I be the incenter of $\triangle ABC$. Let the incircle of $\triangle ABC$ touch sides BC, CA, and AB at points D, E, and F. Let the circumcircle of $\triangle AEF$ meet the circumcircle of $\triangle ABC$ for a second time at point $X \neq A$. Let P denote the intersection of XI and EF. If the product $XP \cdot IP$ can be written as $\frac{m}{n}$ for relatively prime positive integers m, n, find $m + n$.

   Proposed by Sunay Joshi

   Answer: $\boxed{629}$

   We begin by performing a general calculation. Consider a generic triangle $\triangle ABC$. Let D denote the foot of the altitude from A to BC. Let O denote the circumcenter of $\triangle ABC$ and let M denote the midpoint of BC. We will compute the lengths BD, DE, and DO. It is easy to see that $BD = c\cos B = \frac{a^2+c^2-b^2}{2a}$ and $DE = b\cos C = \frac{a^2+b^2-c^2}{2a}$ by the law of cosines. Next, the Pythagorean Theorem applied to $\triangle OMD$ yields $OD^2 = OM^2 + BM^2$, or equivalently $OD^2 = (\frac{a}{2} - c\cos B)^2 + (R^2 - (\frac{a}{2})^2)$. By the law of cosines, $\frac{a}{2} - c\cos B = \frac{a^2}{2a} - \frac{a^2+c^2-b^2}{2a} = \frac{b^2-c^2}{2a}$. Therefore $OD = \sqrt{(\frac{b^2-c^2}{2a})^2 + (R^2 - (\frac{a}{2})^2)}$.

   Returning to the problem, the key fact is that P is the foot of the altitude from D to EF. This can be seen by inverting about the incircle of $\triangle ABC$. Next, by power of a point, the desired product is $XP \cdot IP = PE \cdot PF$. Note that I is the circumcenter of $\triangle DEF$. It therefore suffices to evaluate the distances computed in the first paragraph of this solution, for the triangle $\triangle DEF$. Let x, y, z denote the lengths EF, FD, DE and r denote the inradius of $\triangle ABC$. Note that $x = 2(s-a)\sin\frac{A}{2}$, and similarly for y, z. Note that $\sin\frac{A}{2} = \sqrt{\frac{1-\cos A}{2}} = \sqrt{\frac{a^2-(b-c)^2}{4bc}}$, and similarly for $\sin\frac{B}{2}$, $\sin\frac{C}{2}$.

   We now compute. Note that $a = 7$, $b = 6$, $c = 5$, so that $s = 9$, $s-a = 2$, $s-b = 3$, $s-c = 4$. The area of $\triangle ABC$ is $rs = \sqrt{9 \cdot 2 \cdot 3 \cdot 4}$, so the inradius is $r = 6\sqrt{6}/9 = 2\sqrt{6}/3$. Next, note that $\sin\frac{A}{2} = \sqrt{\frac{7^2-1^2}{4\cdot6\cdot5}} = \sqrt{\frac{2}{5}}$, $\sin\frac{B}{2} = \sqrt{\frac{6^2-2^2}{4\cdot7\cdot5}} = \sqrt{\frac{8}{35}}$, $\sin\frac{C}{2} = \sqrt{\frac{5^2-1^2}{4\cdot6\cdot7}} = \sqrt{\frac{1}{7}}$. Therefore $x = 2\cdot2\cdot\sqrt{\frac{2}{5}}$, $y = 2\cdot3\cdot\sqrt{\frac{8}{35}}$, $z = 2\cdot4\cdot\sqrt{\frac{1}{7}}$, and so $x^2 = \frac{224}{35}$, $y^2 = \frac{288}{35}$, $z^2 = \frac{320}{35}$. $2x = \frac{8\sqrt{10}}{5}$.

Plugging the above into our formulae, we see that $PE = \frac{x^2+z^2-y^2}{2x} = \frac{256/35}{8\sqrt{10}/5}$ and $PF = \frac{x^2+y^2-z^2}{2x} = \frac{192/35}{8\sqrt{10}/5}$. Also, $IP = \sqrt{(\frac{y^2-z^2}{2x})^2 + (r^2 - (\frac{x}{2})^2)} = \sqrt{(\frac{32/35}{8\sqrt{10}/5})^2 + (\frac{8}{3} - \frac{56}{35})} = \sqrt{(\frac{4}{7\sqrt{10}})^2 + (\frac{16}{15})} = \sqrt{\frac{16}{490} + \frac{16}{15}} = \frac{2\sqrt{3030}}{105}$. Therefore the desired product is

$$XP \cdot IP = PE \cdot PF = \frac{256/35}{8\sqrt{10}/5} \cdot \frac{192/35}{8\sqrt{10}/5} = \frac{384}{245}$$

and our final answer is $384 + 245 = 629$.

8. Let $\triangle ABC$ have sidelengths $BC = 7$, $CA = 8$, and, $AB = 9$, and let Ω denote the circumcircle of $\triangle ABC$. Let circles $\omega_A, \omega_B, \omega_C$ be internally tangent to the minor arcs $\widehat{BC}, \widehat{CA}, \widehat{AB}$ of Ω, respectively, and tangent to the segments BC, CA, AB at points X, Y, and, Z, respectively. Suppose that $\frac{BX}{XC} = \frac{CY}{YA} = \frac{AZ}{ZB} = \frac{1}{2}$. Let t_{AB} be the length of the common external tangent of ω_A and ω_B, let t_{BC} be the length of the common external tangent of ω_B and ω_C, and let t_{CA} be the length of the common external tangent of ω_C and ω_A. If $t_{AB} + t_{BC} + t_{CA}$ can be expressed as $\frac{m}{n}$ for relatively prime positive integers m, n, find $m + n$.

   Proposed by Sunay Joshi

   Answer: $\boxed{59}$

   Let $k = \frac{BX}{BC} = \frac{1}{3}$. First, we show that $t_{AB} = (1-k)^2a + k^2b + (1-k)kc$. Let t_A, t_B, t_C denote the length of the tangent from A, B, C to $\omega_A, \omega_B, \omega_C$, respectively. By Casey's Theorem applied to circles $(A), (B), \omega_A, (C)$, we find that $c \cdot CX + b \cdot BX = a \cdot t_A$. Since $CX = (1-k)a$ and $BX = ka$, solving yields $t_A = kb + (1-k)c$. By symmetry we find $t_B = kc + (1-k)a$. Applying Casey's Theorem to circles $(A), (B), \omega_A, \omega_B$, we find $c \cdot t_{AB} + AY \cdot BX = t_A \cdot t_B$. Since $AY = (1-k)b$ and $BX = ka$, solving yields the claimed expression for t_{AB}.

   By symmetry, we therefore have $t_{BC} = (1-k)^2b + k^2c + (1-k)ka$ and $t_{CA} = (1-k)^2c + k^2a + (1-k)kb$. Summing yields $t_{AB} + t_{BC} + t_{CA} = ((1-k)^2 + k^2 + (1-k)k)(a+b+c) = (k^2-k+1)(a+b+c)$. Plugging in $k = \frac{1}{3}$ and $a = 7$, $b = 8$, $c = 9$, we find $\frac{7}{9}\cdot 24 = \frac{56}{3}$ and hence an answer of $56+3 = 59$.