Geometry A Solutions

1. Frist Campus Center is located 1 mile north and 1 mile west of Fine Hall. The area within 5 miles of Fine Hall that is located north and east of Frist can be expressed in the form $\frac{a}{b}\pi - c$, where a, b, c are positive integers and a and b are relatively prime. Find $a + b + c$.

   Proposed by: Zhuo Qun Song

   Answer: $\boxed{30}$

   Create a $2x2$ square about the radius-5 circle centered at Fine. Subtracting this area from the circle gives $25\pi - 4$. By symmetry, the area desired is $\frac{1}{4}$ of this area, for a final answer of $\frac{25}{4}\pi - 1$. So the answer is $25 + 4 + 1 = \boxed{30}$.

2. Let $\overline{AD}$ be a diameter of a circle. Let point B be on the circle, point C be on $\overline{AD}$ such that A, B, C form a right triangle with right angle at C. The value of the hypotenuse of the triangle is 4 times the square root of its area. If $\overline{BC}$ has length 30, what is the length of the radius of the circle?

   Proposed by: Nathan Bergman

   Answer: $\boxed{120}$

   Let h be the length of $\overline{BC}$ and r the radius of the circle. Triangle ABD has one length diameter of the circle, so angle ABD is a right angle and we have similar triangles. Let $AB = y$. Then $AC(AD) = AC(2r) = y^2 = (4\sqrt{\frac{h(AC)}{2}})^2 = 8h(AC)$, so $r = 4h$. We are given h is 30, so the radius of the circle is $\boxed{120}$.

3. Let $\triangle ABC$ satisfy $AB = 17$, $AC = \frac{70}{3}$ and $BC = 19$. Let I be the incenter of $\triangle ABC$ and E be the excenter of $\triangle ABC$ opposite A. (Note: this means that the circle tangent to ray AB beyond B, ray AC beyond C, and side BC is centered at E.) Suppose the circle with diameter IE intersects AB beyond B at D. If $BD = \frac{a}{b}$ where a, b are coprime positive integers, find $a + b$.

   Proposed by: Mel Shu

   Answer: $\boxed{22}$

   An angle chase shows that $BICED$ is cyclic, and the reflection across AI takes D to C. Therefore $AD = AC = \frac{70}{3}$, so $BD = \frac{70}{3} - 17 = \frac{19}{3}$ and $a + b = \boxed{22}$.

4. Triangle ABC has $\angle A = 90°$, $\angle C = 30°$, and $AC = 12$. Let the circumcircle of this triangle be W. Define D to be the point on arc BC not containing A so that $\angle CAD = 60°$. Define points E and F to be the foots of the perpendiculars from D to lines AB and AC, respectively. Let J be the intersection of line EF with W, where J is on the minor arc AC. The line DF intersects W at H other than at D. The area of the triangle FHJ is in the form $\frac{a}{b}(\sqrt{c} - \sqrt{d})$ for positive integers a, b, c, d, where a, b are relatively prime, and the sum of a, b, c, d is minimal. Find $a + b + c + d$.

   Proposed by: Alan Chung

   Answer: $\boxed{23}$

   Let G be the point of intersection of EF with BC. Since G lies on the Simson line with respect to point D, $DG \perp BC$. It follows that $CDGF$ is cyclic because $\angle CFD = 90°$ by construction. Thus, $\angle DCB = \angle BAD = 30°$, so $\angle ACD = 60°$.Since $\angle DAC = 60°$, triangle ACD is equilateral, which implies that $AF = FC = 6$. Since CDF is a $30° - 60° - 90°$ right triangle, $DF = 6\sqrt{3}$. By the power of a point with respect to point F, $AF(CF) = FH(FD) \Rightarrow FH = 2\sqrt{3}$. At this point, it suffices to find length FJ because the area of FJH is given by $\frac{1}{2}FG(FH)\sin(30°)$. We can find FJ through the following lemma.

Lemma: Suppose equilateral triangle ABC is inscribed in a circle with side length 12. Let D be the foot of the altitude from A, let E be the intersection of line AD with the circumcircle other than at A. Let F be the point on minor arc BC, not containing A, so that $\angle EDF = 30°$. Then $EF = 3(\sqrt{5}-1)$.

Proof: Suppose line DF intersects AB at G, and let DF intersect the circle at H, other than at F. We have $\angle ADG = 30°$, so angle $\angle AGH$ is $60°$. Since triangle BDG is equilateral, G is the midpoint of AB. So GH is exactly the same length as DF. Let $DF = HG = x$. $DG = 6$, so by power of a point, $BD \cdot DC = DF \cdot DH$, so $36 = x(x+6)$, which implies that $x = 3(\sqrt{5}-1)$. This concludes the lemma.

Then,

$$[FHJ] = \frac{1}{2}FH(FJ)\sin(30°) = \frac{3}{2}(\sqrt{15}-\sqrt{3}).$$

So the answer is $\boxed{23}$.

5. Let $\triangle ABC$ be triangle with side lengths $AB = 9, BC = 10, CA = 11$. Let O be the circumcenter of $\triangle ABC$. Denote $D = AO \cap BC, E = BO \cap CA, F = CO \cap AB$. If $1/AD + 1/BE + 1/FC$ can be written in simplest form as $\frac{a\sqrt{b}}{c}$, find $a+b+c$.

   Proposed by: Kapil Chandran

   Answer: $\boxed{43}$

   For a general point P inside $\triangle ABC$, we have

   $$DP/DA + EP/EB + FP/FC = [PBC]/[ABC] + [PCA]/[ABC] + [PAB]/[ABC] = 1.$$

   Therefore,

   $$PA/DA + PB/EB + PC/FC = 2.$$

   Now apply in the case $P = O$. We have $R = \frac{33}{4\sqrt{2}}$, giving an answer of $\frac{8\sqrt{2}}{33}$ ($\boxed{43}$).

6. Let triangle ABC have $\angle BAC = 45°$ and circumcircle Γ and let M be the intersection of the angle bisector of $\angle BAC$ with Γ. Let Ω be the circle tangent to segments $\overline{AB}$ and $\overline{AC}$ and internally tangent to Γ at point T. Given that $\angle TMA = 45°$ and that $TM = \sqrt{100-50\sqrt{2}}$, the length of BC can be written as $a\sqrt{b}$, where b is not divisible by the square of any prime. Find $a+b$.

   Proposed by: Kevin Feng

   Answer: $\boxed{12}$

   Let I be the incenter of ABC and L be the antipode of M. It is well-known that T, I, and L are collinear and that $\angle MTL = \angle MTI = 90°$. Thus, $\triangle MTI$ is an isosceles right triangle, so we have $MI = \sqrt{2} \cdot TM = 2\dfrac{\sqrt{50}}{\sqrt{2+\sqrt{2}}}$. It is also well-known that $MI = MB = MC$, so $MB = MC = 2 \cdot \dfrac{\sqrt{50}}{\sqrt{2+\sqrt{2}}}$. Then because $\angle BAC = 45°$, we have $\angle BMC = 180° - 45° = 135°$. Finally, by Law of Cosines, we have $BC^2 = MB^2 + MC^2 - 2 \cdot MB \cdot MC \cos 135° = 2 \cdot (100 + 50\sqrt{2})/(\sqrt{2+\sqrt{2}})^2$, so $BC = 2\sqrt{50} = 10\sqrt{2}$, giving an answer of $\boxed{12}$.

7. Let $ABCD$ be a parallelogram such that $AB = 35$ and $BC = 28$. Suppose that $BD \perp BC$. Let ℓ_1 be the reflection of AC across the angle bisector of $\angle BAD$, and let ℓ_2 be the line through B perpendicular to CD. ℓ_1 and ℓ_2 intersect at a point P. If PD can be expressed in simplest form as $\frac{m}{n}$, find $m+n$.

Proposed by: Mel Shu

Answer: $\boxed{113}$

It is a well-known lemma that given a triangle $\triangle XYZ$, the symmedian from X, the perpendicular bisector of YZ, and the tangents to the circumcircle at Y and Z are all concurrent. Considering $\triangle ABD$, we have that AC is a median, hence AP is a symmedian, and through some angle chasing, we have that BP is a tangent to the circumcircle. It follows by the above lemma that P is the point of concurrency, and so P must lie on the perpendicular bisector of BD and PD is a tangent to the circumcircle of $\triangle ABD$. We note that $\triangle ABD \sim \triangle DPM$ where M is the midpoint of BD, and that both triangles are 3-4-5 triangles. Therefore $PD = PB$, and $\triangle PDM \sim \triangle ABD$, where M is the midpoint of BD, hence $PD = 6\frac{7*20}{4*16} = \frac{7*15}{4*2} = \frac{105}{8}$, so the answer is $105 + 8 = \boxed{113}$.

8. Let ω be a circle. Let E be on ω and S be outside ω such that line segment SE is tangent to ω. Let R be on ω. Let line SR intersect ω at B other than R, such that R is between S and B. Let I be the intersection of the bisector of $\angle ESR$ with the line tangent to ω at R; let A be the intersection of the bisector of $\angle ESR$ with ER. If the radius of the circumcircle of $\triangle EIA$ is 10, the radius of the circumcircle of $\triangle SAB$ is 14, and $SA = 18$, then IA can be expressed in simplest form as $\frac{m}{n}$. Find $m + n$.

   Proposed by: Marko Medvedev

   Answer: $\boxed{97}$

   Triangles $\triangle SER$ and $\triangle SEB$ are similar. Take the point X to be the image of the point I under similarity mapping $\triangle SER \to \triangle SEB$. Now X is on the bisector of angle $\angle ESB$ as well, and it holds that $\angle XEB = \angle IRE = \angle RBE = \angle RES = \angle AES$ hence points X and A are isogonal conjugates in triangle SEB hence $\angle AEI = \angle XBE = \angle ABS$.

   The length is thus $\frac{r}{R} \cdot SA$, where R is the radius of the circumcircle of $\triangle SBA$, and r be the radius of the circumcircle $\triangle EIA$. That is $\frac{10}{14} \cdot 18 = \frac{180}{14}$, giving an answer of $\boxed{97}$.

If you believe that any of these answers is incorrect, or that a problem had multiple reasonable interpretations or was incorrectly stated, you may appeal at http://tinyurl.com/PUMaCappeal2018. All appeals must be in by 1 PM to be considered.