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Combinatorics B Solutions

1. I have a 2 by 4 grid of squares; how many ways can I shade at least one of the squares so that no two shaded squares share an edge?

   Proposed by Austen Mazenko

   Answer: $\boxed{40}$

   We proceed with casework on the number of shaded squares.

   Case 1 (one shaded square): evidently, there are $2 \cdot 4 = 8$ different squares that can be chosen as the shaded one.

   Case 2 (two shaded squares): we complementary count. There are $\binom{8}{2} = 28$ ways to choose two squares to shade. Now, there are 6 pairs with a shared vertical edge, and 4 pairs with a shared horizontal edge, so subtracting off these cases gives $28 - 10 = 18$ options.

   Case 3 (three shaded squares): the shaded squares evidently must all be in different columns. If they are in three consecutive columns, which can happen 2 ways, then evidently a checkerboard pattern is the only way that two of them aren't adjacent, giving $2 \cdot 2$ options. Similarly, if one of the two middle columns lacks a shaded square, which happens 2 ways, then the two shaded squares in adjacent columns must be in different rows; this can happen 2 ways, and the other shaded square also has two choices for row. Thus this case contributes $4 + 2 \cdot 2 \cdot 2 = 12$ in total.

   Case 4 (four shaded squares): they all must be in a checkerboard pattern, of which there are 2, each determined by the location of the shaded square in the leftmost column.

   In sum, there are $8 + 18 + 12 + 2 = 40$ ways to shade.

2. Amir enters Fine Hall and sees the number 2 written on a blackboard. Amir can perform the following operation: he flips a coin, and if it is heads, he replaces the number x on the blackboard with $3x + 1$; otherwise, he replaces x with $\lfloor x/3 \rfloor$. If Amir performs this operation four times, let $\frac{m}{n}$ denote the expected number of times that he writes the digit 1 on the blackboard, where m, n are relatively prime positive integers. Find $m + n$.

   Proposed by Sunay Joshi

   Answer: $\boxed{27}$

   There are only two numbers that can appear in four operations that contain the digit 1, namely 1 and 13. The sequences of flips that contain the digit 1 are: HTTH (one 1), TTTH (one 1), TTHT (one 1), TTHH (one 1), THTT (one 1), THTH (two 1s), THHT (two 1s), and THHH (two 1s). The expected value is therefore $\frac{1}{16}(1+1+1+1+1+2+2+2) = \frac{11}{16}$ and our answer is $11 + 16 = 27$.

3. Alien Connor starts at $(0, 0)$ and walks around on the integer lattice. Specifically, he takes one step of length one in a uniformly random cardinal direction every minute, unless his previous four steps were all in the same direction in which case he randomly picks a new direction to step in. Every time he takes a step, he leaves toxic air on the lattice point he just left, and the toxic cloud remains there for 150 seconds. After taking 5 steps in total, the probability that he has not encountered his own toxic waste can be written as $\frac{a}{b}$ for relatively prime positive integers a, b. Find $a + b$.

   Proposed by Ben Zenker

   Answer: $\boxed{505}$

   Due to parity, we can see that the only way he can encounter his own toxic waste is by walking directly backwards. The toxic waste stays in the air for 2 full step sizes, but disappears after

3, and there's no way to take two more steps and return to where you started.
First, suppose his first four steps are all in the same direction, which happens with probability $\frac{1}{4^3}$. Then, the probability he avoids his own toxic waste with his last step is $\frac{2}{3}$, contributing a probability of $\frac{2}{3} \cdot \frac{1}{4^3}$. Otherwise, we see the probability he makes it four steps without hitting is own toxic waste while also not going the same direction every step is $\left(\frac{3}{4}\right)^3 - \frac{1}{4^3} = \frac{13}{32}$. Conditioned on this, we see the probability his last step also avoids the toxic air is again $\frac{3}{4}$. Thus, our final answer is $\frac{2}{3} \cdot \frac{1}{4^3} + \frac{13}{32} \cdot \frac{3}{4} = \frac{4}{3 \cdot 128} + \frac{117}{3 \cdot 128} = \frac{121}{384}$, giving a final answer of $121 + 384 = 505$.

4. Let $\oplus$ denote the xor binary operation. Define $x \star y = (x + y) - (x \oplus y)$. Compute

$$\sum_{k=1}^{63} (k \star 45).$$

(*Remark:* The xor operator works as follows: when considered in binary, the kth binary digit of $a \oplus b$ is 1 exactly when the kth binary digits of a and b are different. For example, $5 \oplus 12 = 0101_2 \oplus 1100_2 = 1001_2 = 9$.)

Proposed by Julian Shah

Answer: $\boxed{2880}$

Solution 1: Consider pairing the kth term with the $(63 - k)$th term:

$$(k \star 45) + ((63 - k) \star 45) = 63 + 2 \cdot 45 - [k \oplus 45 + (63 - k) \oplus 45]$$

k and $63 - k$ differ in every binary digit, so the values of $k \oplus 45$ and $(63 - k) \oplus 45$ will be completely complementary; hence, they add to 63 (when performing the addition $k \oplus 45$, switching a 0 to a 1 in k's representation will switch the result of that output digit).

So, when we pair k and $63 - k$, the result is $2 \cdot 45$. This means $\sum_{k=0}^{63} (k \star 45) = 64 \cdot 45$, but $(0 \star 45) = 45 - 45 = 0$, so our final answer is actually $64 \cdot 45 = \boxed{2880}$.

Solution 2: For any x and y, we can obtain the relationship: $(x \oplus y) + 2(x \& y) = x + y$, where $x \& y$ is the bitwise-*and* operator: $x \& y$ has a 1 in a place only if both x and y do.
This occurs since $x \oplus y$ returns 0 in all places where x and y are both 1, whereas under normal addition this *should* contribute 2 times that place value. Adding $2(x \& y)$ covers this difference. Therefore, $x \star y = (x + y) - (x \oplus y) = 2(x \& y)$.
Again, note that $((63 - k) \& 45) + (k \& 45) = 45$. Collectively, k and $63 - k$ will 'select' all the 1's places of 45, so adding them will return exactly 45.

Using the same trick as earlier, $\sum_{k=0}^{63} (k \& 45) = 32 \cdot 45$, so our answer is twice this, $2 \cdot 32 \cdot 45 = \boxed{2880}$.

5. The integers from 1 to 25, inclusive, are randomly placed into a 5 by 5 grid such that in each row, the numbers are increasing from left to right. If the columns from left to right are numbered 1, 2, 3, 4, and 5, then the expected column number of the entry 23 can be written as $\frac{a}{b}$ where a and b are relatively prime positive integers. Find $a + b$.

Proposed by Rishi Dange

Answer: $\boxed{17}$

The answer is $\frac{14}{3}$. We proceed using casework, seeing pretty easily that 23 cannot be in columns 1 or 2.

Case 1: 23 is in column 5 In a given row, by putting 23 as the rightmost item, there are now $\binom{22}{4}$ ways to fill in the rest of the row. This makes for a probability of 5 rows $\times \frac{\binom{22}{4}}{\binom{25}{5}}$.

Case 2: 23 is in column 4 The right two elements in the row with 23 are either 23 and 24 or 23 and 25, which makes the "rows" coefficient above $5 \times 2 = 10$. Thus we now have a probability of $10 \times \frac{\binom{22}{3}}{\binom{25}{5}}$.

Case 3: 23 is in column 3 The right three elements must be 23, 24, and 25, so the coefficient is 5 again to represent the number of possible rows. Thus we now have a probability of $5 \times \frac{\binom{22}{2}}{\binom{25}{5}}$.

Computing the expected value using these probabilities (quite a bit of stuff cancels out), we find the expected value to be $\frac{14}{3}$.

6. A sequence of integers $a_1, a_2, \ldots, a_n$ is said to be *sub-Fibonacci* if $a_1 = a_2 = 1$ and $a_i \leq a_{i-1} + a_{i-2}$ for all $3 \leq i \leq n$. How many sub-Fibonacci sequences are there with 10 terms such that the last two terms are both 20?

   Proposed by Daniel Carter

   Answer: $\boxed{238}$

   The number of sequences of length 10 that end in 20, 20 is just the number of sequences of length 9 which end in 20, since it is impossible for it to be the case that $a_8 < 0$ and $a_9 = 20$, as the seventh Fibonacci number (i.e. the maximum possible value for a_7) is only 13.

   Let F_n be the Fibonacci numbers, where $F_1 = F_2 = 1$. Suppose we chose the maximum value $a_{i-1} + a_{i-2}$ for every term a_i in our sequence except for some a_j, which we made k less than the maximum possible value. Then $a_n = F_n - kF_{n-j+1}$. This works similarly if we make multiple terms less than their maximum; if we define $d_i = a_i - a_{i-1} - a_{i-2}$, then we find $a_n = F_n - \sum_{i=3}^{n} d_i F_{n-i+1}$. Since $F_9 = 34$, the question is equivalent to asking for the number of choices of d_i which make $\sum_{i=3}^{9} d_i F_{10-i} = 14$.

   In order to compute this, let's define $f(k, t)$ to be the number of choices of d_i such that $\sum_{i=1}^{t} d_i F_i = k$. By convention, $f(0, t) = 1$ for all t and $f(k, t) = 0$ if k is negative. We are looking for $f(14, 7)$. We have $f(k, t) = f(k, t-1) + f(k - F_t, t)$, i.e. we either stop increasing d_t and move on to smaller t or increment d_t. With this recurrence, we can quickly fill up a table of values for f until we hit $f(14, 7)$, which we find to be 238.

7. There are n assassins numbered from 1 to n, and all assassins are initially alive. The assassins play a game in which they take turns in increasing order of number, with assassin 1 getting the first turn, then assassin 2, etc., with the order repeating after assassin n has gone; if an assassin is dead when their turn comes up, then their turn is skipped and it goes to the next assassin in line. On each assassin's turn, they can choose to either kill the assassin who would otherwise move next or to do nothing. Each assassin will kill on their turn unless the only option for guaranteeing their own survival is to do nothing. If there are 2023 assassins at the start of the game, after an entire round of turns in which no one kills, how many assassins must remain?

   Proposed by Ben Lemkin

   Answer: $\boxed{1023}$

   We show by induction that number of the form $n = 2^k - 1$ are stable, being no one shoots, while all others are not. $n = 1, 2$ are trivial; for $n = 3$, we observe it's stable for the following reason; if person 1 shoots person 2, then person 3 will kill person 1, so to avoid dying person 1 will not shoot; by symmetry, this means none of them will shoot. Indeed, note by symmetry that we only need to assess the first person's behavior, as if person 1 shoots then it's not stable, while if person 1 does not shoot then it is stable. With the base case proven, suppose $n = 2^k - 1$ is stable. Consider $n = 2^k$; the first person will shoot the second person, because then there are $2^k - 1$ people remaining, and by assumption no one will shoot in this scenario. Similarly, for

$n = 2^k + 1$, if the first person shoots the second person, then 2^k people will remain standing, whence the third person will proceed to shoot the fourth person after which $2^k - 1$ people are left and everyone ceases. Thus, the first person will shoot the second person as long as the first person is not also the fourth person (meaning $4 \equiv 1 \pmod{2^k - 1}$). Continuing like so, we see that for $n = 2^k + a$ for $a \geq 0$ that person 1 will shoot, then person 3 will shoot, etc., all the way up to person $2(a+1) - 1$ shooting person $2(a+1)$, after which there are $2^k - 1$ people left and no one else shoots. This sequence of events will hold up until the point when $2(a+1) \equiv 1 \pmod{2^k + a}$, because that would means person 1 gets shot; the smallest a this holds for, and thus the next smallest stable position, is $a = 2^k - 1$, implying $n = 2^{k+1} - 1$. Thus, by induction, stable n are of the form $2^k - 1$, and our answer is 1023.

8. For a positive integer n, let P_n be the set of sequences of $2n$ elements, each 0 or 1, where there are exactly n 1's and n 0's. I choose a sequence uniformly at random from P_n. Then, I partition this sequence into maximal blocks of consecutive 0's and 1's. Define $f(n)$ to be the expected value of the sum of squares of the block lengths of this uniformly random sequence. What is the largest integer value that $f(n)$ can take on?

Proposed by Kevin Ren

Answer: $\boxed{121}$

It's easier to compute the expected value of $\sum \binom{a}{2}$, where the a's are the block lengths. Note that

$$\mathbb{E}\left[\sum a^2\right] = \mathbb{E}\left[\sum 2\binom{a}{2} + a\right] = 2\mathbb{E}\left[\binom{a}{2}\right] + 2n,$$

since the sum of block lengths is clearly $2n$. Hence, it suffices to show $\mathbb{E}\left[\sum \binom{a}{2}\right] = 2 \cdot \frac{n(n-1)}{n+2}$.

Note that $\sum \binom{a}{2}$ equals the number of pairs of indices (i, j) such that (i, j) belong to the same block. We take advantage of this by instead computing the expectation that (i, j) belong to the same block, and sum over (i, j) (this amounts to a change of summation). The probability that (i, j) belong to the same block is $2 \cdot \binom{2n - |j-i+1|}{n} / \binom{2n}{n}$. And then by various binomial identities,

$$\sum_{i<j} 2 \cdot \binom{2n - |j-i+1|}{n} = \sum_{k=2}^{2n} 2(2n-k+1)\binom{2n-k}{n} = \sum_{k=2}^{2n} 2(n+1)\binom{2n-k+1}{n+1}$$

$$= 2(n+1)\binom{2n}{n+2}.$$

Finally, $\mathbb{E}\left[\sum \binom{a}{2}\right] = 2(n+1)\binom{2n}{n+2} / \binom{2n}{n} = 2 \cdot \frac{n(n-1)}{n+2}$, as desired. This gives the actual desired expected value as $\frac{6n^2}{n+2}$. In order for this to be an integer, note that it equals $\frac{6(n-2)(n+2)}{n+2} + \frac{24}{n+2}$, which is an integer precisely when $n+2 | 24$. It is evidently maximized for $n = 22$, giving an answer of $\frac{6 \cdot 22^2}{24} = 121$.