Combinatorics B

1. You have four fair 6-sided dice, each numbered from 1 to 6 (inclusive). If all four dice are rolled, the probability that the product of the rolled numbers is prime can be written as $\frac{a}{b}$, where a, b are relatively prime. What is $a + b$?

2. There are five dots arranged in a line from left to right. Each of the dots is colored from one of five colors so that no 3 consecutive dots are all the same color. How many ways are there to color the dots?

3. In an election between A and B, during the counting of the votes, neither candidate was more than 2 votes ahead, and the vote ended in a tie, 6 votes to 6 votes. Two votes for the same candidate are indistinguishable. In how many orders could the votes have been counted? One possibility is AABBABBABABA.

4. Let N be the number of sequences of natural numbers $d_1, d_2, ..., d_{10}$ such that the following conditions hold: $d_1 | d_2, ..., d_9 | d_{10}$ and $d_{10} | 6^{2018}$. Evaluate the remainder when N is divided by 2017.

5. Alex starts at the origin O of a hexagonal lattice. Every second, he moves to one of the six vertices adjacent to the vertex he is currently at. If he ends up at X after 2018 moves, then let p be the probability that the shortest walk from O to X (where a valid move is from a vertex to an adjacent vertex) has length 2018. Then p can be expressed as $\frac{a^m - b}{c^n}$, where a, b, and c are positive integers less than 10; a and c are not perfect squares; and m and n are positive integers less than 10000. Find $a + b + c + m + n$.

6. If a and b are selected uniformly from $\{0, 1, ..., 511\}$ with replacement, the expected number of 1's in the binary representation of $a + b$ can be written in simplest form as $\frac{m}{n}$. Compute $m + n$.

7. How many ways are there to color the 8 regions of a three-set Venn Diagram with 3 colors such that each color is used at least once? Two colorings are considered the same if one can be reached from the other by rotation and reflection.

8. Frankie the Frog starts his morning at the origin in $\mathbb{R}^2$. He decides to go on a leisurely stroll, consisting of $3^1 + 3^{10} + 3^{11} + 3^{100} + 3^{101} + 3^{110} + 3^{111} + 3^{1000}$ moves, starting with the 1st move. On the nth move, he hops a distance of
$$\max\{k \in \mathbb{Z} : 3^k | n\} + 1,$$
then turns $90°$ degrees counterclockwise. What is the square of the distance from his final position to the origin?