Algebra B Solutions

1. Let q be the sum of the expressions $a_1^{-a_2^{a_3^{a_4}}}$ over all permutations (a_1, a_2, a_3, a_4) of $(1, 2, 3, 4)$. Determine $\lfloor q \rfloor$.

   Proposed by Frank Lu

   Answer: $\boxed{8}$

   We perform casework on the position of the 1. If $a_1 = 1$, then we obtain a contribution of $3! \cdot 1 = 6$. If $a_2 = 1$, then we obtain a contribution of $2! \cdot (\frac{1}{2} + \frac{1}{3} + \frac{1}{4}) = 2 + \frac{1}{6}$. If $a_3 = 1$, then the contribution is $\frac{1}{2^3} + \frac{1}{2^4} + \frac{1}{3^2} + \frac{1}{3^4} + \frac{1}{4^2} + \frac{1}{4^3}$, which is bounded by $\frac{1}{2} = \frac{1}{8} + \frac{1}{16} + \frac{1}{8} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16}$. Finally, if $a_4 = 1$, then the contribution is bounded by $6 \cdot 2^{-81}$, which is less than $\frac{1}{6}$, say. Therefore $q \geq 8$ and $q \leq 8 + \frac{1}{6} + \frac{1}{2} + \frac{1}{6} < 9$. Our answer is thus $\lfloor q \rfloor = 8$.

2. A pair (f, g) of degree 2 real polynomials is called *foolish* if $f(g(x)) = f(x) \cdot g(x)$ for all real x. How many positive integers less than 2023 can be a root of $g(x)$ for some foolish pair (f, g)?

   Proposed by Austen Mazenko

   Answer: $\boxed{2021}$

   We claim that if (f, g) is foolish, then there exist real numbers a, b such that $f(x) = ax(x + b)$ and $g(x) = x^2 + bx - b$. To see this, let r be a root of g, and plug $x = r$ into the functional equation to find $f(0) = 0$. This immediately implies that $f(x) = ax(x+b)$ for some a, b. Next, plug this form of f into the functional equation to find $ag(x)(g(x) + b) = ax(x + b)g(x)$. Since $\deg g = 2$, g is not identically zero, hence $g(x) + b = x(x + b)$. Rearranging yields the claim.

   Now, note that a positive integer x is a root of $x^2 + bx - b$ iff $b = \frac{x^2}{1-x}$. It follows that any $x \neq 1$ is a root of some g. Hence the valid positive integers between 1 and 2022 inclusive are all numbers except 1. This yields an answer of $2022 - 1 = 2021$ integers.

3. Given two polynomials f and g satisfying $f(x) \geq g(x)$ for all real x, a *separating line* between f and g is a line $h(x) = mx + k$ such that $f(x) \geq h(x) \geq g(x)$ for all real x. Consider the set of all possible separating lines between $f(x) = x^2 - 2x + 5$ and $g(x) = 1 - x^2$. The set of slopes of these lines is a closed interval $[a, b]$. Determine $a^4 + b^4$.

   Proposed by Frank Lu

   Answer: $\boxed{184}$

   Solution: We consider $y = mx + b$ for our line. To have $f(x) \geq mx + b$, we need $x^2 - (m + 2)x + 5 - b$ to have discriminant at most 0. This becomes the condition $b \leq 5 - (m + 2)^2/4$. Similarly, for the other polynomial, we need $b \geq 1 + m^2/4$. Thus, the set of possible values of m are $1 + m^2/4 \leq 5 - (m + 2)^2/4$. In other words, we need $m^2/2 + m - 3 \leq 0$. Thus, our values for a and b are the roots of this polynomial (which we rewrite as $m^2 + 2m - 6$). To get $a^4 + b^4$, we write this as $(a^2 + b^2)^2 - 2a^2b^2 = ((a + b)^2 - 2ab)^2 - 2(ab)^2$. This is then $(2^2 + 12)^2 + 2 \cdot 6^2 = 256 - 72 = 184$.

4. Let $P(x, y)$ be a polynomial with real coefficients in the variables x, y that is not identically zero. Suppose that $P(\lfloor 2a \rfloor, \lfloor 3a \rfloor) = 0$ for all real numbers a. If P has the minimum possible degree and the coefficient of the monomial y is 4, find the coefficient of x^2y^2 in P.
   (The *degree* of a monomial $x^m y^n$ is $m + n$. The *degree* of a polynomial $P(x, y)$ is then the maximum degree of any of its monomials.)

   Proposed by Sunay Joshi

   Answer: $\boxed{216}$

Note that the possible values for the pair $(\lfloor 2x \rfloor, \lfloor 3x \rfloor)$ are $(2k, 3k), (2k, 3k+1), (2k+1, 3k+1), (2k+1, 3k+2)$ for $k \in \mathbb{Z}$. These are roots of the linear polynomials $3x - 2y$, $3x - 2y + 2$, $3x - 2y - 1$, and $3x - 2y + 1$, respectively. It follows that $P(x, y)$ is divisible by the product $(3x - 2y)(3x - 2y + 2)(3x - 2y - 1)(3x - 2y + 1)$. Letting $z = 3x - 2y$, the product equals $z(z + 2)(z^2 - 1) = z^4 + 2z^3 - z^2 - 2z$. The coefficient of y is given as $-2(-2) = 4$, hence in fact $P(x, y)$ equals the product. To find the coefficient of x^2y^2, apply the Binomial Theorem to find $\binom{4}{2} \cdot 3^2 \cdot (-2)^2 = 216$, our answer.

5. Find the number of real solutions (x, y) to the system of equations:

$$\begin{cases} \sin(x^2 - y) = 0 \\ |x| + |y| = 2\pi \end{cases}$$

Proposed by Ben Zenker

Answer: $\boxed{52}$

Note that $\sin(x^2 - y) = 0$ iff $x^2 - y = k\pi$ for some $k \in \mathbb{Z}$. Therefore we seek the number of intersections of the parabola $y = x^2 - k\pi$ with the square $|x| + |y| = 2\pi$ for each k.

Since the vertex of the parabola has y-coordinate $-\pi k$, it is clear that there are 0 intersections for $k \leq -3$ and 1 intersection for $k = -2$.

If the vertex of the parabola lies strictly within the square, it is clear that there must be exactly be 2 intersections. This occurs for $-1 \leq k \leq 1$.

When $k = 2$, the vertex of the parabola is the vertex $(0, -2\pi)$ of the square, and one can check that there are 5 intersections, including the vertex.

For $k \geq 13$, there are no intersections, since the x-intercept of the parabola equals $x = \sqrt{\pi k} > 2\pi$. For $3 \leq k \leq 12$, it is easy to see that there are 4 intersections.

Summing, we find a total of $1 + 2 \cdot 3 + 5 + 10 \cdot 4 = 52$ intersections, our answer.

6. The set C of all complex numbers z satisfying $(z + 1)^2 = az$ for some $a \in [-10, 3]$ is the union of two curves intersecting at a single point in the complex plane. If the sum of the lengths of these two curves is ℓ, find $\lfloor \ell \rfloor$.

Proposed by Julian Shah

Answer: $\boxed{16}$

We want solutions to $z^2 + (2 - a)z + 1 = 0$. The discriminant is non-negative when $a \in (-\infty, 0] \cup [4, \infty)$, so for our purposes, $a \leq 0$. When the discriminant is non-negative, it can be seen that the solutions lie between the solutions to $x^2 + (2 - (-10))z + 1$; this interval has length $2\sqrt{35}$.

The remaining values of a are in $(0, 3]$. The solutions when $a \in (0, 3]$ are non-real, so they must be conjugates, and they are reciprocals, so it follows that they lie on the unit circle. Furthermore, they're real part is equal to $\frac{-(2-a)}{2}$, which ranges from -1 to $\frac{1}{2}$; thus, the solution set here is the portion of the unit circle with real part less than $\frac{1}{2}$, which comprises two thirds of the unit circle. Thus, the length of this region is $\frac{4\pi}{3}$.

The desired length is then the sum of the lengths of these two regions, which is $2\sqrt{35} + \frac{4\pi}{3}$. Rewriting, this is $\sqrt{140} + \frac{4\pi}{3}$, which has floor 16.

7. Suppose that x, y, z are nonnegative real numbers satisfying the equation

$$\sqrt{xyz} - \sqrt{(1 - x)(1 - y)z} - \sqrt{(1 - x)y(1 - z)} - \sqrt{x(1 - y)(1 - z)} = -\frac{1}{2}.$$

The largest possible value of $\sqrt{xy}$ equals $\frac{a+\sqrt{b}}{c}$, where a, b, and c are positive integers such that b is not divisible by the square of any prime. Find $a^2+b^2+c^2$.

Proposed by Frank Lu

Answer: $\boxed{29}$

We first observe that x, y, z are required to be real numbers between 0 and 1. With this in mind, this suggests the parametrization by $x = \cos^2\alpha_1, y = \cos^2\alpha_2$, and $z = \cos^2\alpha_3$, where the values of $\cos\alpha_1, \cos\alpha_2, \cos\alpha_3$ lie between 0 and $\frac{\pi}{2}$.

This means that, substituting in the values, we get the equation $\cos\alpha_1\cos\alpha_2\cos\alpha_3 - \sin\alpha_1\sin\alpha_2\cos\alpha_3 - \sin\alpha_1\cos\alpha_2\sin\alpha_3 - \cos\alpha_1\sin\alpha_2\sin\alpha_3$. But we can apply the sum of angles formula to yield that this is equal to $\cos(\alpha_1+\alpha_2)\cos\alpha_3 - \sin(\alpha_1+\alpha_2)\sin\alpha_3 = \cos(\alpha_1+\alpha_2+\alpha_3)$. It follows that $\alpha_1+\alpha_2+\alpha_3$ is equal to $\frac{2\pi}{3}$.

However, notice that $\sqrt{xy} = \cos\alpha_1\cos\alpha_2 = \frac{1}{2}(\cos(\alpha_1+\alpha_2)+\cos(\alpha_1-\alpha_2))$. From here, notice that given α_3, we can maximize this value by making $\alpha_1 = \alpha_2$. It then suffices to find the α_3 such that $\frac{1}{2}(\cos(\alpha_1+\alpha_2)+1)$ is maximized. But to do this, we need to minimize $\alpha_1+\alpha_2$.

We recall, on the other hand, that $\alpha_3 \leq \frac{\pi}{2}$, meaning that we need to have $\alpha_1+\alpha_3 \geq \frac{\pi}{6}$. Using this value gives us our maximum value as $\frac{2+\sqrt{3}}{4}$. The answer that we seek is then $2^2+3^2+4^2 = 4+9+16 = 29$.

8. Let x, y, z be positive real numbers satisfying $4x^2-2xy+y^2 = 64$, $y^2-3yz+3z^2 = 36$, and $4x^2+3z^2 = 49$. If the maximum possible value of $2xy+yz-4zx$ can be expressed as $\sqrt{n}$ for some positive integer n, find n.

   Proposed by Sunay Joshi

   Answer: $\boxed{2205}$

   Consider the substitution $a = 2x$, $b = y$, $c = z\sqrt{3}$. The system of equations becomes $a^2+b^2-ab = 8^2$, $b^2+c^2-bc\sqrt{3} = 6^2$, and $c^2+a^2 = 7^2$. The desired quantity becomes $ab+bc\frac{1}{\sqrt{3}} - ca\frac{2}{\sqrt{3}} = \frac{4}{\sqrt{3}}(\frac{1}{2}ab\frac{\sqrt{3}}{2} + \frac{1}{2}bc\frac{1}{2} - ca\frac{1}{2})$. By the Law of Cosines, the values a, b, c can be interpreted geometrically as follows. Consider a quadrilateral $ABCD$ with $AB = a$, $AC = b$, $AD = c$, $\angle BAC = 60°$, and $\angle CAD = 30°$. Then the given equalities imply that $BC = 8$, $CD = 6$, and $BD = 7$. By the sine area formula, the desired quantity can be seen to equal $\frac{4}{\sqrt{3}}([BAC]+[DAC]-[BAD])$.

   We now distinguish two configurations: (1) if A, C lie on the same side of line BD, and (2) if A, C lie on opposite sides of line BD. In either case, the absolute value of the desired quantity is $\frac{4}{\sqrt{3}}[BCD]$, and configuration (2) attains the positive (hence maximum) value. Since the sides of $\triangle BCD$ are 6, 7, 8, Heron's formula implies that $[BCD] = \frac{21\sqrt{15}}{4}$. Hence our quantity is $\frac{4}{\sqrt{3}} \cdot \frac{21\sqrt{15}}{4} = 21\sqrt{5} = \sqrt{2205}$, and our answer is 2205.