P U M ∴ C

Algebra B

1. Let q be the sum of the expressions $a_1^{-a_2^{a_3^{a_4}}}$ over all permutations (a_1, a_2, a_3, a_4) of $(1, 2, 3, 4)$. Determine $\lfloor q \rfloor$.

2. A pair (f, g) of degree 2 real polynomials is called *foolish* if $f(g(x)) = f(x) \cdot g(x)$ for all real x. How many positive integers less than 2023 can be a root of $g(x)$ for some foolish pair (f, g)?

3. Given two polynomials f and g satisfying $f(x) \geq g(x)$ for all real x, a *separating line* between f and g is a line $h(x) = mx + k$ such that $f(x) \geq h(x) \geq g(x)$ for all real x. Consider the set of all possible separating lines between $f(x) = x^2 - 2x + 5$ and $g(x) = 1 - x^2$. The set of slopes of these lines is a closed interval $[a, b]$. Determine $a^4 + b^4$.

4. Let $P(x, y)$ be a polynomial with real coefficients in the variables x, y that is not identically zero. Suppose that $P(\lfloor 2a \rfloor, \lfloor 3a \rfloor) = 0$ for all real numbers a. If P has the minimum possible degree and the coefficient of the monomial y is 4, find the coefficient of x^2y^2 in P.
(The *degree* of a monomial $x^m y^n$ is $m + n$. The *degree* of a polynomial $P(x, y)$ is then the maximum degree of any of its monomials.)

5. Find the number of real solutions (x, y) to the system of equations:
$$\begin{cases} \sin(x^2 - y) = 0 \\ |x| + |y| = 2\pi \end{cases}$$

6. The set C of all complex numbers z satisfying $(z + 1)^2 = az$ for some $a \in [-10, 3]$ is the union of two curves intersecting at a single point in the complex plane. If the sum of the lengths of these two curves is ℓ, find $\lfloor \ell \rfloor$.

7. Suppose that x, y, z are nonnegative real numbers satisfying the equation
$$\sqrt{xyz} - \sqrt{(1-x)(1-y)z} - \sqrt{(1-x)y(1-z)} - \sqrt{x(1-y)(1-z)} = -\frac{1}{2}.$$
The largest possible value of $\sqrt{xy}$ equals $\frac{a+\sqrt{b}}{c}$, where a, b, and c are positive integers such that b is not divisible by the square of any prime. Find $a^2 + b^2 + c^2$.

8. Let x, y, z be positive real numbers satisfying $4x^2 - 2xy + y^2 = 64$, $y^2 - 3yz + 3z^2 = 36$, and $4x^2 + 3z^2 = 49$. If the maximum possible value of $2xy + yz - 4zx$ can be expressed as $\sqrt{n}$ for some positive integer n, find n.

Name:

Team:

Write answers in table below:

Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8