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Algebra A

1. Let a, b, c, d, e, f be real numbers such that $a^2+b^2+c^2 = 14$, $d^2+e^2+f^2 = 77$, and $ad+be+cf = 32$. Find $(bf - ce)^2 + (cd - af)^2 + (ae - bd)^2$.

2. If θ is the unique solution in $(0, \pi)$ to the equation $2\sin(x)+3\sin\left(\frac{3x}{2}\right)+\sin(2x)+3\sin\left(\frac{5x}{2}\right) = 0$, then $\cos(\theta) = \frac{a-\sqrt{b}}{c}$ for positive integers a, b, c such that a and c are relatively prime. Find $a + b + c$.

3. Let $P(x)$ be a polynomial with integer coefficients satisfying
$$(x^2 + 1)P(x - 1) = (x^2 - 10x + 26)P(x)$$
for all real numbers x. Find the sum of all possible values of $P(0)$ between 1 and 5000, inclusive.

4. The set of real values of a such that the equation $x^4 - 3ax^3 + (2a^2 + 4a)x^2 - 5a^2x + 3a^2$ has exactly two nonreal solutions is the set of real numbers between x and y, where $x < y$. If $x + y$ can be written as $\frac{m}{n}$ for relatively prime positive integers m, n, find $m + n$.

5. Compute $\left\lfloor \sum_{k=0}^{10} \left(3 + 2\cos\left(\frac{2\pi k}{11}\right)\right)^{10} \right\rfloor \pmod{100}$.

6. A polynomial $p(x) = \sum_{j=1}^{2n-1} a_j x^j$ with real coefficients is called *mountainous* if $n \geq 2$ and there exists a real number k such that the polynomial's coefficients satisfy $a_1 = 1$, $a_{j+1} - a_j = k$ for $1 \leq j \leq n - 1$, and $a_{j+1} - a_j = -k$ for $n \leq j \leq 2n - 2$; we call k the *step size* of $p(x)$. A real number k is called *good* if there exists a mountainous polynomial $p(x)$ with step size k such that $p(-3) = 0$. Let S be the sum of all good numbers k satisfying $k \geq 5$ or $k \leq 3$. If $S = \frac{b}{c}$ for relatively prime positive integers b, c, find $b + c$.

7. Let S be the set of degree 4 polynomials f with complex number coefficients satisfying $f(1) = f(2)^2 = f(3)^3 = f(4)^4 = f(5)^5 = 1$. Find the mean of the fifth powers of the constant terms of all the members of S.

8. Given a positive integer m, define the polynomial
$$P_m(z) = z^4 - \frac{2m^2}{m^2+1}z^3 + \frac{3m^2-2}{m^2+1}z^2 - \frac{2m^2}{m^2+1}z + 1.$$
Let S be the set of roots of the polynomial $P_5(z) \cdot P_7(z) \cdot P_8(z) \cdot P_{18}(z)$. Let w be the point in the complex plane which minimizes $\sum_{z \in S} |z - w|$. The value of $\sum_{z \in S} |z - w|^2$ equals $\frac{a}{b}$ for relatively prime positive integers a and b. Compute $a + b$.

Name:

Team:

Write answers in table below:

Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8