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Algebra A Solutions

1. Let a, b, c, d, e, f be real numbers such that $a^2+b^2+c^2 = 14$, $d^2+e^2+f^2 = 77$, and $ad+be+cf = 32$. Find $(bf - ce)^2 + (cd - af)^2 + (ae - bd)^2$.

   Proposed by Sunay Joshi

   Answer: $\boxed{54}$

   Solution: Let $u = (a, b, c)$, $v = (d, e, f)$ be vectors in $\mathbb{R}^3$. Then the identity $|u \times v|^2 = |u|^2|v|^2 - (u \cdot v)^2$ implies that the desired expression is simply $(a^2+b^2+c^2)(d^2+e^2+f^2) - (ad+be+cf)^2$. This evaluates to $14 \cdot 77 - 32^2 = 54$.

2. If θ is the unique solution in $(0, \pi)$ to the equation $2\sin(x)+3\sin\left(\frac{3x}{2}\right)+\sin(2x)+3\sin\left(\frac{5x}{2}\right) = 0$, then $\cos(\theta) = \frac{a-\sqrt{b}}{c}$ for positive integers a, b, c such that a and c are relatively prime. Find $a + b + c$.

   Proposed by Ben Zenker and Nancy Xu

   Answer: $\boxed{110}$

   Using sum-to-product, we get $\sin\left(\frac{3x}{2}\right) + \sin\left(\frac{5x}{2}\right) = 2\sin(2x)\cos\left(\frac{x}{2}\right)$.
   Factor out a $\sin(x)$ of the whole expression (after using double angle on $\sin(2x)$), to get:

   $$\sin(x)\left(2 + 2\cos(x) + 12\cos(x)\cos\left(\frac{x}{2}\right)\right) = 0$$

   $\sin(x) > 0$ in $(0, \pi)$, so we can safely ignore it. Let $u = \cos\left(\frac{x}{2}\right)$, then $\cos(x) = 2u^2 - 1$ using double angle. We now solve $2+2(2u^2-1)+12u(2u^2-1) = 0$, which becomes $6u^3+u^2-3u = 0$. The solution $u = 0$ corresponds to $x = \pi$, so we ignore it as well.
   We then just need the solution to $6u^2 + u - 3 = 0$, which is $u = \frac{-1+\sqrt{73}}{12}$.
   Compute $\cos(x) = 2u^2 - 1 = \frac{1-\sqrt{73}}{36}$, so $a + b + c = 1 + 73 + 36 = \boxed{110}$.

3. Let $P(x)$ be a polynomial with integer coefficients satisfying

   $$(x^2 + 1)P(x - 1) = (x^2 - 10x + 26)P(x)$$

   for all real numbers x. Find the sum of all possible values of $P(0)$ between 1 and 5000, inclusive.

   Proposed by Sunay Joshi

   Answer: $\boxed{5100}$

   It is clear that the only constant solution is $P \equiv 0$, for which $P(0)$ is not in the desired range. Therefore we assume P is nonconstant in what follows. Note that since the functional equation holds for all reals, it holds for all complex numbers. Next, note that the roots of $x^2 + 1$ are $\pm i$, while the roots of $x^2 - 10x + 26$ are $\pm i + 5$. Plugging in $x = i$, we find $P(i) = 0$. Plugging in $x = i + 1$, we find $P(i + 1) = 0$. Plugging in $x = i + 2$, we find $P(i + 3) = 0$. Lastly, plugging in $x = i + 3$, we find $P(i + 4) = 0$. Since P has real coefficients, its roots also include the conjugates $-i, -i + 1, -i + 2, -i + 3, -i + 4$. Therefore $P(x)$ can be written as $P(x) = Q(x)(x^2 + 1)(x^2 - 2x + 2)(x^2 - 4x + 5)(x^2 - 6x + 10)(x^2 - 8x + 17)$. We now claim that $Q(x)$ is a nonzero constant. Plugging our expression for P into our functional equation, we find $Q(x - 1) = Q(x)$ for all x, hence $Q(x) \equiv c \neq 0$ is a constant.

   To finish, set $x = 0$ to find $P(x) = 1700c$. The only integer multiples of 1700 between 1 and 5000 are 1700 and 3400, hence our answer is $1700 + 3400 = 5100$.

4. The set of real values of a such that the equation $x^4 - 3ax^3 + (2a^2 + 4a)x^2 - 5a^2x + 3a^2$ has exactly two nonreal solutions is the set of real numbers between x and y, where $x < y$. If $x+y$ can be written as $\frac{m}{n}$ for relatively prime positive integers m, n, find $m+n$.

Proposed by Frank Lu

Answer: $\boxed{8}$

First, we consider trying to factor this into quadratics. Notice that this equals

$$x^4 - 3tx^3 + (2t^2 - 2t)x + t^2x - 3t^2 = (x^2 - tx + t)(x^2 - 2tx + 3t).$$

Therefore, to have two nonreal solutions, one of the discriminants of the quadratics needs to be negative, and the other is nonnegative. In particular, it follows that we need $t^2 - 4t < 0$ and $4t^2 - 12t \geq 0$ or $t^2 - 4t \geq 0$ and $4t^2 - 12t < 0$. For the former to hold, notice that we need $0 < t < 4$, but $t > 3$. The latter cannot hold, however: $t^2 - 4t \geq 0$ implies that $t \geq 4$ or $t \leq 0$, but $4t^2 - 12t < 0$ implies that $0 < t < 3$. Therefore, we see that $a = 3, b = 4$, and $a + b = 7 = 7/1$. Our answer is thus $7 + 1 = 8$.

5. Compute $\left\lfloor \sum_{k=0}^{10} \left(3 + 2\cos\left(\frac{2\pi k}{11}\right)\right)^{10} \right\rfloor \pmod{100}$.

Proposed by Sunay Joshi and Ben Zenker

Answer: $\boxed{91}$

Let $n = 10$. We claim that the sum equals

$$(n+1)\sum_{k=0}^{\lfloor n/2 \rfloor} 3^{n-2k}\binom{n}{2k}\binom{2k}{k} \tag{1}$$

Let $\omega = \exp(2\pi i/(n+1))$. The summand is $(\omega^k + \omega^{-k} + 3)^n$, which by the multinomial expansion equals $\sum_{a+b+c=n}\binom{n}{a,b,c}3^c\omega^{k(a-b)}$. Since $0 \leq |a-b| < n+1$, $\sum_{k=0}^{n} = (n+1)\mathbf{1}_{a=b}$. Therefore the sum becomes

$$(n+1)\sum_{a+b+c=n}\binom{n}{a,b,c}3^{n-a-b}\mathbf{1}_{a=b} = (n+1)\sum_{a=0}^{n}3^{n-2a}\binom{n}{a,a,n-2a} \tag{2}$$

$$= (n+1)\sum_{a=0}^{\lfloor n/2 \rfloor}3^{n-2a}\frac{n!}{a!a!(n-2a)!} \tag{3}$$

$$= (n+1)\sum_{a=0}^{\lfloor n/2 \rfloor}3^{n-2a}\binom{n}{2a}\binom{2a}{a}, \tag{4}$$

as claimed.

The desired remainder is therefore

$$11 \cdot \left[3^{10}\binom{10}{0}\binom{0}{0} + 3^8\binom{10}{2}\binom{2}{1} + 3^6\binom{10}{4}\binom{4}{2} + 3^4\binom{10}{6}\binom{6}{3} + 3^2\binom{10}{8}\binom{8}{4} + 3^0\binom{10}{10}\binom{10}{5}\right] \tag{5}$$

$$\equiv 11 \cdot \left[3^{10} + 3^8 \cdot 90 + 3^6 \cdot 10 \cdot 6 + 3^4 \cdot 10 \cdot 20 + 3^2 \cdot 45 \cdot 70 + 52\right] \tag{6}$$

$$\equiv 91 \pmod{100} \tag{7}$$

6. A polynomial $p(x) = \sum_{j=1}^{2n-1} a_j x^j$ with real coefficients is called *mountainous* if $n \geq 2$ and there exists a real number k such that the polynomial's coefficients satisfy $a_1 = 1$, $a_{j+1} - a_j = k$ for

$1 \leq j \leq n-1$, and $a_{j+1} - a_j = -k$ for $n \leq j \leq 2n-2$; we call k the *step size* of $p(x)$. A real number k is called *good* if there exists a mountainous polynomial $p(x)$ with step size k such that $p(-3) = 0$. Let S be the sum of all good numbers k satisfying $k \geq 5$ or $k \leq 3$. If $S = \frac{b}{c}$ for relatively prime positive integers b, c, find $b + c$.

Proposed by Sunay Joshi

Answer: $\boxed{101}$

We claim that the only good values of k are $k = \frac{7}{3}$ and $\frac{61}{12}$, corresponding to $n = 2$ and $n = 3$ respectively. This yields $S = \frac{89}{12}$ and an answer of 101.

To see this, note that a generic mountainous polynomial $p(x)$ can be written as

$$p(x) = (1-k)\frac{x^{2n} - x}{x - 1} + kx\frac{(x^n - 1)^2}{(x-1)^2}$$

if $x \neq 1$. This follows from the observation that $\frac{x^{2n}-x}{x-1} = x + x^2 + \ldots + x^{2n-1}$ and $\frac{(x^n-1)^2}{(x-1)^2} = (x^{n-1} + x^{n-2} + \ldots + 1)^2 = x + 2x^2 + \ldots + nx^n + (n-1)x^{n+1} + \ldots + x^{2n-2}$. Hence $p(x) = 0$ implies that $(1-k)\frac{x^{2n}-x}{x-1} + kx\frac{(x^n-1)^2}{(x-1)^2} = 0$. Rearranging and solving for k, we find

$$k = 1 - \frac{x^n + \frac{1}{x^n} - 2}{x^{n-1} + \frac{1}{x^{n-1}} - 2}$$

As $n \to \infty$, $k = k(n)$ tends to $1 - x$. In our case $x = -3$, so the limit equals 4. It follows that there are only finitely many n such that $|k - 4| \geq 1$. Calculating $k(n)$ for $n = 2, 3, 4$, we find $k(2) = 7/3$, $k(3) = 61/12$.

We claim that for $n \geq 4$, $|k(n) - 4| < 1$, so that $n = 2, 3$ are the only valid cases. Note that

$$|k(n) - 4| = \left| \frac{8 + \frac{8}{(-3)^n}}{(-3)^{n-1} + \frac{1}{(-3)^{n-1}} - 2} \right|$$

We split into the cases when n is even ($n \geq 4$) and n is odd ($n \geq 5$).

If n is even, then

$$|k(n) - 4| = \frac{8 + \frac{8}{3^n}}{3^{n-1} + \frac{1}{3^{n-1}} + 2}$$

The inequality $|k(n) - 4| < 1$ is equivalent to $\frac{1}{3}3^{2n} - 6 \cdot 3^n - 5 > 0$, i.e. $\frac{1}{3}x^2 - 6x - 5 > 0$ for $x \geq 81$, which is true.

If n is odd, then

$$|k(n) - 4| = \frac{8 - \frac{8}{3^n}}{3^{n-1} + \frac{1}{3^{n-1}} - 2}$$

The inequality $|k(n) - 4| < 1$ is equivalent to $\frac{1}{3}3^{2n} - 10 \cdot 3^n + 11 > 0$, i.e. $\frac{1}{3}x^2 - 10x + 11 > 0$ for $x \geq 243$, which is true. The result follows.

7. Let S be the set of degree 4 polynomials f with complex number coefficients satisfying $f(1) = f(2)^2 = f(3)^3 = f(4)^4 = f(5)^5 = 1$. Find the mean of the fifth powers of the constant terms of all the members of S.

Proposed by Michael Cheng

Answer: $\boxed{1643751}$

Let $N = 5$ for convenience. By the given condition, $f(n) = \zeta_n$ for $1 \leq n \leq N$, where ζ_n is an n-th root of unity. Since f is a degree $N - 1$ polynomial, the Lagrange interpolation formula implies that $f(x) = \sum_{n=1}^N f(n) \prod_{m \neq n} \frac{x-m}{n-m}$, where the product runs over $m \in \{1, \ldots, N\}$, $m \neq n$. We desire the constant term of f, namely $f(0) = \sum_{n=1}^N f(n) \prod_{m \neq n} \frac{-m}{n-m}$. Note that $\prod_{m \neq n} \frac{m}{n-m} = \frac{(-1)\cdots(-(n-1))}{(n-1)\cdots(1)} \cdot \frac{(n+1)\cdots N}{1\cdots(N-n)} = (-1)^{n-1}\binom{N}{n}$. Let $r_n := (-1)^{n-1}\binom{N}{n}$, so that $f(0) = \sum_{n=1}^N \zeta_n r_n$.

We now consider $f(0)^M$, where $M = 5$ for convenience. Expand the power to obtain

$$f(0)^M = \sum_{|\alpha|=M} \zeta_1^{\alpha_1} \cdots \zeta_N^{\alpha_N} \cdot r_1^{\alpha_1} \cdots r_N^{\alpha_N} \cdot \binom{M}{\alpha} \tag{8}$$

Here the sum runs over all N-tuples $\alpha = (\alpha_1, \ldots, \alpha_N)$ of nonnegative integers satisfying $\sum_{n=1}^N \alpha_n = M$, and the multinomial coefficient $\binom{M}{\alpha} := \frac{M!}{\alpha_1! \cdots \alpha_N!}$ counts the number of ways a given summand occurs. Note that averaging over all possible f is equivalent to averaging over all possible N-tuples $(\zeta_1, \ldots, \zeta_N)$. Therefore if a given α is such that n does not divide α_n for some $1 \leq n \leq N$, then $\sum_{\zeta_n} \zeta_n^{\alpha_n} = 0$ (where the sum runs over all n-th roots of unity ζ_n), hence α contributes zero to the average. In other words, the only N-tuples α that contribute to the average are those for which n divides α_n for all $1 \leq n \leq N$; and further the contribution of such an α is simply $r_1^{\alpha_1} \cdots r_N^{\alpha_N} \cdot \binom{M}{\alpha}$. Call these N-tuples *good*. We enumerate such good N-tuples, using the fact that $N = 5$ and $M = 5$. The partitions of $M = 5$ are: 5, $4+1$, $3+2$, $3+1+1$, $2+2+1$, $2+1+1+1$, and $1+1+1+1+1$. Note that for any positive integer d, a good tuple cannot have more than $\tau(d)$ indices n for which $\alpha_n | d$, where τ denotes the number of divisors of d. Applying this fact to $d = 1$ and $d = 2$ eliminates the fourth, fifth, and sixth partitions above. The only valid partitions are 5, $4+1$, and $3+2$.

The partition 5 can correspond to two good tuples: α with $\alpha_1 = 5$ and $\alpha_n = 0$ for $n \neq 1$; or α with $\alpha_5 = 5$ and $\alpha_n = 0$ for $n \neq 5$. By our formula above, these contribute $(r_1^5 + r_5^5)\binom{5}{5}$ to the average.

The partition $4+1$ can correspond to two good tuples: α with $\alpha_1 = 1$, $\alpha_2 = 4$, and $\alpha_n = 0$ otherwise; or α with $\alpha_1 = 1$, $\alpha_4 = 4$, and $\alpha_n = 0$ otherwise. By our formula above, these contribute $(r_1^1 r_2^4 + r_1^1 r_4^4)\binom{5}{4}$ to the average.

The partition $3+2$ can correspond to three good tuples: α with $\alpha_1 = 2$, $\alpha_3 = 3$, and $\alpha_n = 0$ otherwise; α with $\alpha_1 = 3$, $\alpha_2 = 2$, and $\alpha_n = 0$ otherwise; or α with $\alpha_2 = 2$, $\alpha_3 = 3$, and $\alpha_n = 0$ otherwise. By our formula above, these contribute $(r_1^2 r_3^3 + r_1^3 r_2^2 + r_2^2 r_3^3)\binom{5}{3}$ to the average.

Therefore our answer is

$$(r_1^5 + r_5^5)\binom{5}{5} + (r_1^1 r_2^4 + r_1^1 r_4^4)\binom{5}{4} + (r_1^2 r_3^3 + r_1^3 r_2^2 + r_2^2 r_3^3)\binom{5}{3} \tag{9}$$

where $r_n = (-1)^{n-1}\binom{5}{n}$ implies $r_1 = 5$, $r_2 = -10$, $r_3 = 10$, $r_4 = -5$, and $r_5 = 1$. Plugging in yields the answer of 1643751, as desired.

8. Given a positive integer m, define the polynomial

$$P_m(z) = z^4 - \frac{2m^2}{m^2+1} z^3 + \frac{3m^2-2}{m^2+1} z^2 - \frac{2m^2}{m^2+1} z + 1.$$

Let S be the set of roots of the polynomial $P_5(z) \cdot P_7(z) \cdot P_8(z) \cdot P_{18}(z)$. Let w be the point in the complex plane which minimizes $\sum_{z \in S} |z - w|$. The value of $\sum_{z \in S} |z - w|^2$ equals $\frac{a}{b}$ for relatively prime positive integers a and b. Compute $a + b$.

Proposed by Owen Yang and Atharva Pathak

Answer: $\boxed{171}$

We claim that $w = \frac{1}{2}$. To show this, we prove that the roots of P_m come in pairs (z_1, z_2), (z_3, z_4) on the unit circle such that $z_1, z_2, \frac{1}{2}$ are collinear and such that $z_3, z_4, \frac{1}{2}$ are collinear. By the triangle inequality any minimizer w of the sum of distances must lie on the lines $z_1 z_2$ and $z_3 z_4$, so that $w = \frac{1}{2}$.

We now prove these claims. Note that the coefficients of P_m are symmetric, so that $\frac{1}{z^2} P_m$ can be regarded as a polynomial in $z + \frac{1}{z}$. Applying this trick and rescaling by z^2, we obtain the factorization

$$P_m(z) = (z^2 - \frac{m}{m+i} z + \frac{m-i}{m+i})(z^2 - \frac{m}{m-i} z + \frac{m+i}{m-i}) \tag{10}$$

The roots z_1, z_2 of the first factor are given as

$$z = \frac{m \pm \sqrt{m^2 - 4(m^2+1)}}{2(m+i)} = \frac{m \pm i\sqrt{3m^2+4}}{2(m+i)} \tag{11}$$

so that

$$z - \frac{1}{2} = \frac{-i \pm i\sqrt{3m^2+4}}{2(m+i)} = \frac{i}{2(m+i)}(-1 \pm \sqrt{3m^2+4}) \tag{12}$$

with ratio $\frac{-1+\sqrt{3m^2+4}}{-1-\sqrt{3m^2+4}} \in \mathbb{R}$, implying the collinearity of $z_1, z_2, \frac{1}{2}$. Further, note that the modulus of z_1, z_2 are given as

$$|z|^2 = \frac{|m \pm i\sqrt{3m^2+4}|^2}{|2(m+i)|^2} = \frac{m^2 + (3m^2+4)}{4(m+1)^2} = 1 \tag{13}$$

implying that z_1, z_2 lie on the unit circle. Since the second quadratic factor is obtained by conjugating the first, we obtain the same results for the remaining roots z_3, z_4. The above claims follow, so that $w = \frac{1}{2}$.

It remains to compute $\sum |z - \frac{1}{2}|^2$, where z runs over the roots of P_5, P_7, P_8, P_{18}. Let z be a root of $P_m(z)$. Then $|z - \frac{1}{2}|^2 = (z - \frac{1}{2})(\bar{z} - \frac{1}{2}) = \frac{5}{4} - \frac{1}{2}(z + \frac{1}{z})$, since $|z| = 1$. By Vieta, $\sum z = \sum \frac{1}{z} = \frac{2m^2}{m^2+1}$, where the sum runs over all four roots of P_m, and where we used the fact that the coefficients of P_m are symmetric. Therefore P_m contributes $5 - \frac{2m^2}{m^2+1} = 3 + \frac{2}{m^2+1}$ to the desired sum. Summing over $m \in \{5, 7, 8, 18\}$, we find

$$3 \cdot 4 + 2(\frac{1}{5^2+1} + \frac{1}{7^2+1} + \frac{1}{8^2+1} + \frac{1}{18^2+1}) = 12 + 2(\frac{1}{26} + \frac{1}{50} + \frac{1}{65} + \frac{1}{325}) \tag{14}$$

$$= 12 + \frac{2}{13} \tag{15}$$

$$= \frac{158}{13} \tag{16}$$

so that $a + b = 158 + 13 = 171$, our answer.