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Algebra A Solutions

1. Given two polynomials f and g satisfying $f(x) \geq g(x)$ for all real x, a *separating line* between f and g is a line $h(x) = mx + k$ such that $f(x) \geq h(x) \geq g(x)$ for all real x. Consider the set of all possible separating lines between $f(x) = x^2 - 2x + 5$ and $g(x) = 1 - x^2$. The set of slopes of these lines is a closed interval $[a, b]$. Determine $a^4 + b^4$.

   Proposed by Frank Lu

   Answer: $\boxed{184}$

   Solution: We consider $y = mx + b$ for our line. To have $f(x) \geq mx + b$, we need $x^2 - (m + 2)x + 5 - b$ to have discriminant at most 0. This becomes the condition $b \leq 5 - (m + 2)^2/4$. Similarly, for the other polynomial, we need $b \geq 1 + m^2/4$. Thus, the set of possible values of m are $1 + m^2/4 \leq 5 - (m + 2)^2/4$. In other words, we need $m^2/2 + m - 3 \leq 0$. Thus, our values for a and b are the roots of this polynomial (which we rewrite as $m^2 + 2m - 6$). To get $a^4 + b^4$, we write this as $(a^2 + b^2)^2 - 2a^2b^2 = ((a + b)^2 - 2ab)^2 - 2(ab)^2$. This is then $(2^2 + 12)^2 + 2 \cdot 6^2 = 256 - 72 = 184$.

2. Let $P(x, y)$ be a polynomial with real coefficients in the variables x, y that is not identically zero. Suppose that $P(\lfloor 2a \rfloor, \lfloor 3a \rfloor) = 0$ for all real numbers a. If P has the minimum possible degree and the coefficient of the monomial y is 4, find the coefficient of x^2y^2 in P.
   (The *degree* of a monomial $x^m y^n$ is $m + n$. The *degree* of a polynomial $P(x, y)$ is then the maximum degree of any of its monomials.)

   Proposed by Sunay Joshi

   Answer: $\boxed{216}$

   Note that the possible values for the pair $(\lfloor 2x \rfloor, \lfloor 3x \rfloor)$ are $(2k, 3k), (2k, 3k + 1), (2k + 1, 3k + 1), (2k + 1, 3k + 2)$ for $k \in \mathbb{Z}$. These are roots of the linear polynomials $3x - 2y$, $3x - 2y + 2$, $3x - 2y - 1$, and $3x - 2y + 1$, respectively. It follows that $P(x, y)$ is divisible by the product $(3x - 2y)(3x - 2y + 2)(3x - 2y - 1)(3x - 2y + 1)$. Letting $z = 3x - 2y$, the product equals $z(z + 2)(z^2 - 1) = z^4 + 2z^3 - z^2 - 2z$. The coefficient of y is given as $-2(-2) = 4$, hence in fact $P(x, y)$ equals the product. To find the coefficient of x^2y^2, apply the Binomial Theorem to find $\binom{4}{2} \cdot 3^2 \cdot (-2)^2 = 216$, our answer.

3. Find the number of real solutions (x, y) to the system of equations:

$$\begin{cases} \sin(x^2 - y) = 0 \\ |x| + |y| = 2\pi \end{cases}$$

   Proposed by Ben Zenker

   Answer: $\boxed{52}$

   Note that $\sin(x^2 - y) = 0$ iff $x^2 - y = k\pi$ for some $k \in \mathbb{Z}$. Therefore we seek the number of intersections of the parabola $y = x^2 - k\pi$ with the square $|x| + |y| = 2\pi$ for each k.

   Since the vertex of the parabola has y-coordinate $-\pi k$, it is clear that there are 0 intersections for $k \leq -3$ and 1 intersection for $k = -2$.

   If the vertex of the parabola lies strictly within the square, it is clear that there must be exactly be 2 intersections. This occurs for $-1 \leq k \leq 1$.

   When $k = 2$, the vertex of the parabola is the vertex $(0, -2\pi)$ of the square, and one can check that there are 5 intersections, including the vertex.

For $k \geq 13$, there are no intersections, since the x-intercept of the parabola equals $x = \sqrt{\pi k} > 2\pi$. For $3 \leq k \leq 12$, it is easy to see that there are 4 intersections.

Summing, we find a total of $1 + 2 \cdot 3 + 5 + 10 \cdot 4 = 52$ intersections, our answer.

4. The set C of all complex numbers z satisfying $(z+1)^2 = az$ for some $a \in [-10, 3]$ is the union of two curves intersecting at a single point in the complex plane. If the sum of the lengths of these two curves is ℓ, find $\lfloor \ell \rfloor$.

   Proposed by Julian Shah

   Answer: $\boxed{16}$

   We want solutions to $z^2 + (2-a)z + 1 = 0$. The discriminant is non-negative when $a \in (-\infty, 0] \cup [4, \infty)$, so for our purposes, $a \leq 0$. When the discriminant is non-negative, it can be seen that the solutions lie between the solutions to $x^2 + (2-(-10))z + 1$; this interval has length $2\sqrt{35}$.

   The remaining values of a are in $(0, 3]$. The solutions when $a \in (0, 3]$ are non-real, so they must be conjugates, and they are reciprocals, so it follows that they lie on the unit circle. Furthermore, they're real part is equal to $\frac{-(2-a)}{2}$, which ranges from -1 to $\frac{1}{2}$; thus, the solution set here is the portion of the unit circle with real part less than $\frac{1}{2}$, which comprises two thirds of the unit circle. Thus, the length of this region is $\frac{4\pi}{3}$.

   The desired length is then the sum of the lengths of these two regions, which is $2\sqrt{35} + \frac{4\pi}{3}$. Rewriting, this is $\sqrt{140} + \frac{4\pi}{3}$, which has floor 16.

5. Suppose that x, y, z are nonnegative real numbers satisfying the equation

$$\sqrt{xyz} - \sqrt{(1-x)(1-y)z} - \sqrt{(1-x)y(1-z)} - \sqrt{x(1-y)(1-z)} = -\frac{1}{2}.$$

   The largest possible value of $\sqrt{xy}$ equals $\frac{a+\sqrt{b}}{c}$, where a, b, and c are positive integers such that b is not divisible by the square of any prime. Find $a^2 + b^2 + c^2$.

   Proposed by Frank Lu

   Answer: $\boxed{29}$

   We first observe that x, y, z are required to be real numbers between 0 and 1. With this in mind, this suggests the parametrization by $x = \cos^2 \alpha_1, y = \cos^2 \alpha_2$, and $z = \cos^2 \alpha_3$, where the values of $\cos \alpha_1, \cos \alpha_2, \cos \alpha_3$ lie between 0 and $\frac{\pi}{2}$.

   This means that, substituting in the values, we get the equation $\cos \alpha_1 \cos \alpha_2 \cos \alpha_3 - \sin \alpha_1 \sin \alpha_2 \cos \alpha_3 - \sin \alpha_1 \cos \alpha_2 \sin \alpha_3 - \cos \alpha_1 \sin \alpha_2 \sin \alpha_3$. But we can apply the sum of angles formula to yield that this is equal to $\cos(\alpha_1 + \alpha_2) \cos \alpha_3 - \sin(\alpha_1 + \alpha_2) \sin \alpha_3 = \cos(\alpha_1 + \alpha_2 + \alpha_3)$. It follows that $\alpha_1 + \alpha_2 + \alpha_3$ is equal to $\frac{2\pi}{3}$.

   However, notice that $\sqrt{xy} = \cos \alpha_1 \cos \alpha_2 = \frac{1}{2}(\cos(\alpha_1 + \alpha_2) + \cos(\alpha_1 - \alpha_2))$. From here, notice that given α_3, we can maximize this value by making $\alpha_1 = \alpha_2$. It then suffices to find the α_3 such that $\frac{1}{2}(\cos(\alpha_1 + \alpha_2) + 1)$ is maximized. But to do this, we need to minimize $\alpha_1 + \alpha_2$.

   We recall, on the other hand, that $\alpha_3 \leq \frac{\pi}{2}$, meaning that we need to have $\alpha_1 + \alpha_3 \geq \frac{\pi}{6}$. Using this value gives us our maximum value as $\frac{2+\sqrt{3}}{4}$. The answer that we seek is then $2^2 + 3^2 + 4^2 = 4 + 9 + 16 = 29$.

6. Let x, y, z be positive real numbers satisfying $4x^2 - 2xy + y^2 = 64$, $y^2 - 3yz + 3z^2 = 36$, and $4x^2 + 3z^2 = 49$. If the maximum possible value of $2xy + yz - 4zx$ can be expressed as $\sqrt{n}$ for some positive integer n, find n.

   Proposed by Sunay Joshi

Answer: $\boxed{2205}$

Consider the substitution $a = 2x$, $b = y$, $c = z\sqrt{3}$. The system of equations becomes $a^2 + b^2 - ab = 8^2$, $b^2 + c^2 - bc\sqrt{3} = 6^2$, and $c^2 + a^2 = 7^2$. The desired quantity becomes $ab + bc\frac{1}{\sqrt{3}} - ca\frac{2}{\sqrt{3}} = \frac{4}{\sqrt{3}}(\frac{1}{2}ab\frac{\sqrt{3}}{2} + \frac{1}{2}bc\frac{1}{2} - ca\frac{1}{2})$. By the Law of Cosines, the values a, b, c can be interpreted geometrically as follows. Consider a quadrilateral $ABCD$ with $AB = a$, $AC = b$, $AD = c$, $\angle BAC = 60°$, and $\angle CAD = 30°$. Then the given equalities imply that $BC = 8$, $CD = 6$, and $BD = 7$. By the sine area formula, the desired quantity can be seen to equal $\frac{4}{\sqrt{3}}([BAC] + [DAC] - [BAD])$.

We now distinguish two configurations: (1) if A, C lie on the same side of line BD, and (2) if A, C lie on opposite sides of line BD. In either case, the absolute value of the desired quantity is $\frac{4}{\sqrt{3}}[BCD]$, and configuration (2) attains the positive (hence maximum) value. Since the sides of $\triangle BCD$ are 6, 7, 8, Heron's formula implies that $[BCD] = \frac{21\sqrt{15}}{4}$. Hence our quantity is $\frac{4}{\sqrt{3}} \cdot \frac{21\sqrt{15}}{4} = 21\sqrt{5} = \sqrt{2205}$, and our answer is 2205.

7. For a positive integer $n \geq 1$, let $a_n = \lfloor \sqrt[3]{n} + \frac{1}{2} \rfloor$. Given a positive integer $N \geq 1$, let $\mathcal{F}_N$ denote the set of positive integers $n \geq 1$ such that $a_n \leq N$. Let $S_N = \sum_{n \in \mathcal{F}_N} \frac{1}{a_n^2}$. As N goes to infinity, the quantity $S_N - 3N$ tends to $\frac{a\pi^2}{b}$ for relatively prime positive integers a, b. Given that $\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$, find $a + b$.

   Proposed by Sunay Joshi

   Answer: $\boxed{97}$

   We claim that the desired limit equals $\frac{1}{16}\sum_{k=1}^{\infty}\frac{1}{k^2}$, or equivalently $\frac{1\pi^2}{96}$, which yields an answer of 97.

   Note that $a_n = k$ iff $k \leq \sqrt[3]{n} + \frac{1}{2} < k + 1$, or equivalently $(k - \frac{1}{2})^3 \leq n < (k + \frac{1}{2})^3$. Expanding, we find

$$k^3 - \frac{3}{2}k^2 + \frac{3}{4}k - \frac{1}{8} \leq n < k^3 + \frac{3}{2}k^2 + \frac{3}{4}k + \frac{1}{8}$$

   The upper and lower bounds differ by $3k^2 + \frac{1}{4}$. Note that if m is a real number with $\{m\} > \frac{1}{4}$, then there are exactly $3k^2$ integers n in the interval $[m - (3k^2 + \frac{1}{4}), m)$. However if $\{m\} \in (0, \frac{1}{4}]$, there are exactly $3k^2 + 1$ integers in the interval. The upper bound of $k^3 + \frac{3}{2}k^2 + \frac{3}{4}k + \frac{1}{8}$ has fractional part that is a multiple of $\frac{1}{8}$. Thus there are $3k^2 + 1$ values of n iff the fractional part is exactly $\frac{1}{8}$, namely when $\frac{3}{2}k^2 + \frac{3}{4}k$ is an integer and when k is divisible by 4. It follows that there are $3k^2$ values of n such that $a_n = k$ if 4 does not divide k and $3k^2 + 1$ values otherwise.

   We may therefore rewrite the sum E_N as

$$E_N = \sum_{k=1}^{N} \frac{3k^2 + \mathbf{1}_{[4|k]}}{k^2} - 3N = \sum_{4|k, k \leq N} \frac{1}{k^2} = \frac{1}{16}\sum_{k=1}^{\lfloor N/4 \rfloor} \frac{1}{k^2}$$

   Sending $N \to \infty$, we find a limit of $\pi^2/96$, as desired.

8. The function f sends sequences to sequences in the following way: given a sequence $\{a_n\}_{n=0}^{\infty}$ of real numbers, f sends $\{a_n\}_{n=0}^{\infty}$ to the sequence $\{b_n\}_{n=0}^{\infty}$, where $b_n = \sum_{k=0}^{n} a_k \binom{n}{k}$ for all $n \geq 0$. Let $\{F_n\}_{n=0}^{\infty}$ be the Fibonacci sequence, defined by $F_0 = 0$, $F_1 = 1$, and $F_{n+2} = F_{n+1} + F_n$ for all $n \geq 0$. Let $\{c_n\}_{n=0}^{\infty}$ denote the sequence obtained by applying the function f to the sequence $\{F_n\}_{n=0}^{\infty}$ 2022 times. Find $c_5 \pmod{1000}$.

   Proposed by Sunay Joshi

Answer: $\boxed{775}$

Suppose that a_n satisfies the recurrence $a_{n+2} = sa_{n+1} - pa_n$ for all $n \geq 0$ with $a_0 = 0$, $a_1 = 1$. We claim that if f is applied to $\{a_n\}$, the resulting sequence b_n satisfies the recurrence $b_{n+2} = (s+2)b_{n+1} - (s+p+1)b_n$ for all $n \geq 0$, with $b_0 = 0$, $b_1 = 1$. To see this, suppose that a_n has explicit formula $a_n = c_1\alpha^n + c_2\beta^n$ for $n \geq 0$, where $\alpha + \beta = s$ and $\alpha\beta = p$. Then by definition, $b_n = \sum_{k=0}^{n}(c_1\alpha^k + c_2\beta^k)\binom{n}{k} = c_1\sum_{k=0}^{n}\alpha^k\binom{n}{k} + c_2\sum_{k=0}^{n}\beta^k\binom{n}{k}$. By the Binomial Theorem, this may be rewritten as $b_n = c_1(1+\alpha)^n + c_2(1+\beta)^n$. Thus the recurrence $b_{n+2} = s'b_{n+1} - p'b_n$ satisfies $s' = (1+\alpha)+(1+\beta) = s+2$ and $p' = (1+\alpha)(1+\beta) = 1+(\alpha+\beta)+\alpha\beta = s+p+1$, as claimed. The initial conditions $b_0 = 0$ and $b_1 = 1$ follow from definition: $b_0 = \binom{0}{0}a_0 = 0$ and $b_1 = \binom{1}{0}a_0 + \binom{1}{1}a_1 = 1$.

Since $F_{n+2} = 1F_{n+1} - (-1)F_n$, the Fibonacci sequence has $(s,p) = (1,-1)$. If f is applied k times to $\{F_n\}$ $(k \geq 0)$, one can show by induction that the resulting pair (s,p) is $(s,p) = (2k+1, k^2+k-1)$. In particular, for $k = 2022$, we have $(s,p) \equiv (45, 505) \pmod{1000}$, so that $c_{n+2} \equiv 45c_{n+1} - 505c_n$. We now simply compute the first 6 terms of the sequence $\{c_n\}$: $c_0 = 0$, $c_1 = 1$, $c_2 = 45$, $c_3 = 45^2 - 505 \equiv -408$, $c_4 \equiv 45(-408) - 45(505) \equiv -325$, and $c_5 \equiv 45(-325) - 505(-408) \equiv 775$. Thus our answer is 775.