P U M :. C

Combinatorics A Solutions

1. Joey is playing with a 2-by-2-by-2 Rubik's cube made up of 8 1-by-1-by-1 cubes (with two of these smaller cubes along each of the sides of the bigger cubes). Each face of the Rubik's cube is distinct color. However, one day, Joey accidentally breaks the cube! He decides to put the cube back together into its solved state, placing each of the pieces one by one. However, due to the nature of the cube, he is only able to put in a cube if it is adjacent to a cube he already placed. If different orderings of the ways he chooses the cubes are considered distinct, determine the number of ways he can reassemble the cube.

   Proposed by Frank Lu

   Answer: $\boxed{8640}$.

   Solution: We have 8 choices for the first cube that Joey picks up. Then, he has 3 choices for the second cube and 4 for the third cube, yielding us 96 ways to first construct an L made up of three cubes. Now, note that there are 4 places to put the fourth cube. If Joey decides to not place the cube on top of the center of the L, then we can observe that each of the 4 remaining spots can be filled in any order. Otherwise, if Joey places the cube on the top of the center of the L, he has 3 places for the next cube, and then he can fix the cube with the last 3 pieces in any order. This yields $96 \cdot (3 \cdot 24 + 18) = 96 \cdot 90 = 8640$ ways that Joey can fix the cube.

2. Cary has six distinct coins in a jar. Occasionally, he takes out three of the coins and adds a dot to each of them. Determine the number of orders in which Cary can choose the coins so that, eventually, for each number $i \in \{0, 1, \ldots, 5\}$, some coin has exactly i dots on it.

   Proposed by Frank Lu

   Answer: $\boxed{79200}$.

   Solution: Label the coins $0, 1, \ldots, 5$ by how many dots they end up with; notice that there are 720 ways to make this assignment (depending on how to assign our 6 coins to these dots). Note that, since the sum of the number of dots in total is 15, and we add 3 dots per draw, Cary pulled out coins 5 times. This also means that Cary drew the 5 coin every time. Without loss of generality, assume the 1 coin was drawn in the first pile, and we multiply by 5 later. Now, we have $1\_5, \_\_5, \_\_5, \_\_5, \_\_5$ for our draws. Observe that if the 4 coin is never drawn with the 1 coin, then we have $\binom{5}{2}$ ways to arrange the 2 and 3 coin, all of which work. Otherwise, we have 4 choices for which draw has neither a 1 nor a 4, resulting in an order like the following (multiplying by 4 later): $145, 235, \_45, \_45, \_45$. Here, we have $\binom{3}{1}$ ways to determine the remaining place in which the 2 coin was drawn. Our total is thus $720 \cdot 5 \cdot (10 + 4 \cdot 3) = 79200$.

   Note: We initially had 110 as the answer, but this is incorrect since we stated that we had distinct coins. We apologize for the confusion this would have caused.

3. Katie has a chocolate bar that is a 5-by-5 grid of square pieces, but she only wants to eat the center piece. To get to it, she performs the following operations:

   i. Take a gridline on the chocolate bar, and split the bar along the line.

   ii. Remove the piece that doesn't contain the center.

   iii. With the remaining bar, repeat steps 1 and 2.

   Determine the number of ways that Katie can perform this sequence of operations so that eventually she ends up with just the center piece.

   Proposed by Frank Lu

   Answer: $\boxed{6384}$.

Solution: Note that each sequence of operations is uniquely determined by which line Katie breaks along at each step, so we consider sequences of lines. Label the horizontal lines from top to bottom l_1, l_2, l_3, l_4, and the lines from left to right m_1, m_2, m_3, m_4. Since Katie ends up with the center piece, the four lines that bound the center, l_2, l_3, m_2, m_3 must have all been broken along. Observe that if l_1 was also broken along, it would have had to been before l_2, as no portion of l_1 exists on the same side of l_2 as the center piece. A similar logic holds for l_4, m_1, m_4 with l_3, m_2, m_3, respectively. Note that beyond this restriction, however, every sequence of these lines is a valid sequence of breaks (we can imagine as though Katie makes knife cuts through the whole bar first before taking out just the center piece). If Katie makes i cuts, $4 \leq i \leq 8$, then we have $\binom{4}{i-4}$ ways to pick which of the four lines that don't bound the center have cuts made. Then, of $i!$ ways to arrange these lines, we divide by 2^{i-4} to account for the fact that there is only one allowed relative ordering between an outer line and its corresponding inner line. This yields the following sum:

$$1 \cdot 4! + 4 \cdot \frac{5!}{2} + 6 \cdot \frac{6!}{4} + 4 \cdot \frac{7!}{8} + \frac{8!}{16} = 24 + 240 + 1080 + 2520 + 2520 = 264 + 5040 + 1080 = \boxed{6384}.$$

4. Let $\mathcal{P}$ be the power set of $\{1, 2, 3, 4\}$ (meaning the elements of $\mathcal{P}$ are the subsets of $\{1, 2, 3, 4\}$). How many subsets S of $\mathcal{P}$ are there such that no two distinct integers $a, b \in \{1, 2, 3, 4\}$ appear together in exactly one element of S?

   Proposed by Austen Mazenko

   Answer: $\boxed{21056}$.

   Solution: First, notice that whether or not $\emptyset, \{1\}, \{2\}, \{3\}, \{4\}$ are in S does not affect the pairing condition, so we multiply by 2^5 at the end to account for all possible cases where only some of these are in S.

   Now suppose $\{1, 2, 3, 4\} \in S$. Thus, every pair of elements $a, b \in \{1, 2, 3, 4\}$ appears together in at least one element of S, so they must appear in another. If S has at least two elements of cardinality 3, then this condition is satisfied. There are 11 ways to assign at least two such elements to S, then 2^6 ways to determine which sets of cardinality 2 are elements of S, giving $11 \cdot 2^6$ in this case.

   If S has at least three elements of cardinality 3, then this condition is satisfied. There are 5 ways to assign at least three such elements to S, then 2^6 ways to determine which sets of cardinality 2 are elements of S, giving $5 \cdot 2^6$ in this case. If it has two elements of cardinality 3, WLOG they're 1,2,3 and 2,3,4: note this may be picked 6 ways. Then, 1,4 must be in S while the remaining five sets of cardinality 2 can be assigned in 2^5 ways, giving $6 \cdot 2^5$ possibilities.

   If it has only one, WLOG it's $\{1, 2, 3\}$. Thus, we need $\{1, 4\}, \{2, 4\}, \{3, 4\}$ to all be elements of S. It doesn't matter if the remaining three sets of cardinality 2 are in S, so we have 2^3 ways to assign them; multiplying by 4 to account for the WLOG assumption gives 2^5. If S has no elements of cardinality 3, then every possible set of cardinality 2 must be in S, giving 1 case. Otherwise, $\{1, 2, 3, 4\} \notin S$, so we do casework on the number of three-element sets that are elements of S.

   If all four are elements of S, then each pair of integers occurs in at least two of them, so we may arbitrarily assign the sets of cardinality 2 in 2^6 ways.

   If only three are elements of S, we may choose them 4 ways. Then, WLOG $\{1, 2, 3\} \notin S$, so $\{1, 2\}, \{1, 3\}, \{2, 3\} \in S$ and we may decide to add the other cardinality 2 sets into S in 2^3 ways, giving $4 \cdot 2^3 = 2^5$ in this case.

   If only two are elements of S, we may choose them 6 ways. Then, only a single pair will have occurred twice, so we may either include the two-element subset with this pair or not, giving $2 \cdot 6 = 12$ total cases.

If only one is an element of S, which may be chosen 4 ways, then all the others are fixed.

If none are elements of S, then none of the two-element subsets may be elements of S giving 1 case in this situation.

Tallying our above count gives $5 \cdot 2^6 + 6 \cdot 2^5 + 2^5 + 1 + 2^6 + 2^5 + 12 + 4 + 1 = 658$, which multiplied by 2^5 gives 21056.

5. Jacob has a piece of bread shaped like a figure 8, marked into sections and all initially connected as one piece of bread. The central part of the "8" is a single section, and each of the two loops of "8" is divided into an additional 1010 pieces. For each section, there is a 50 percent chance that Jacob will decide to cut it out and give it to a friend, and this is done independently for each section. The remaining sections of bread form some number of connected pieces. If E is the expected number of these pieces, and k is the smallest positive integer so that $2^k(E - \lfloor E \rfloor) \geq 1$, find $\lfloor E \rfloor + k$. (Here, we say that if Jacob donates all pieces, there are 0 pieces left).

   Proposed by Frank Lu

   Answer: $\boxed{1515}$.

   Solution: Let $n = 1010$ for convenience. We compute the sum $\sum_{k=0}^{n} c_k$, where c_k is the number of ways for Jacob to cut out the pieces to form k pieces. We divide this into two cases.

   First, if the middle piece is taken, notice that this can be viewed as having two "rows." In this case, suppose that we have a pieces from one loop, and b connected pieces from the other loop. Then, notice that, along this row, we can split it up by considering the number of sections taken, going counterclockwise, from the the central (taken) piece to the first remaining piece, and so on. This can be viewed as some equation $x_1 + x_2 + \cdots + x_{2a+1} = n$, where the $x_i \geq 1$, save for x_1 and x_{2a+1}. We see that the number of solutions for this is $\binom{n-2a+1+2a}{2a} = \binom{n+1}{2a}$. Similarly, we see that for b this is $\binom{n+1}{2b}$.

   For the other case, if we have the middle piece, suppose that we have a other pieces not with the middle on one loop, and b on the other. We see that we have now two equations, again. On one hand, we have $x_0 + x_1 + x_2 + \cdots + x_{2a+1} + x_{2a+2} = n$, which again has $\binom{n+1}{2a+2}$ solutions to it. However, there is one slight issue here: notice that if we take $a = 0$, notice that we have another valid solution, namely with just $x_1 = 0$ (namely that the entire loop is taken). Similarly, we have $\binom{n+1}{2b+2}$ solutions in this case, where $b \neq 0$, and for $b = 0$ we have $\binom{n+1}{2} + 1$.

   Notice that our expected value is thus

$$\sum_{a=0}^{n}\sum_{b=0}^{n}(a+b)\binom{n+1}{2a}\binom{n+1}{2b} + \sum_{a=1}^{n}\sum_{b=1}^{n}(a+b+1)\binom{n+1}{2b+2}\binom{n+1}{2a+2} +$$
$$\sum_{b=1}^{n}(b+1)\left(\binom{n+1}{2}+1\right)\binom{n+1}{2b+2} + \sum_{a=1}^{n}(a+1)\left(\binom{n+1}{2}+1\right)\binom{n+1}{2a+2} + \left(\binom{n+1}{2}+1\right)^2,$$

   where we set the "invalid" binomial coefficients to just be 0. But notice that we can write this sum as just

$$\sum_{a=0}^{n}\sum_{b=0}^{n}(a+b)\binom{n+1}{2a}\binom{n+1}{2b} + \sum_{a=0}^{n}\sum_{b=0}^{n}(a+b+1)\binom{n+1}{2b+2}\binom{n+1}{2a+2}$$
$$+ \sum_{b=1}^{n}(b+1)\binom{n+1}{2b+2} + \sum_{a=1}^{n}(a+1)\binom{n+1}{2a+2} + 2\binom{n+1}{2} + 1.$$

We can then further re-write this then as

$$\frac{(n+1)}{2}\sum_{a=0}^{n}\sum_{b=0}^{n}\left(\binom{n}{2a-1}\binom{n+1}{2b}+\binom{n+1}{2a}\binom{n}{2b-1}\right)+\frac{(n+1)}{2}\sum_{a=1}^{n}\sum_{b=1}^{n}\left(\binom{n}{2a-1}\binom{n+1}{2b}\right.$$
$$\left.+\binom{n+1}{2a}\binom{n}{2b-1}\right)-\sum_{a=1}^{n}\sum_{b=1}^{n}\binom{n+1}{2b}\binom{n+1}{2a}+2\sum_{b=1}^{n}\left(\frac{n+1}{2}\binom{n}{2b+1}\right)+2\binom{n+1}{2}+1.$$

Finally, noticing that $\sum\limits_{a=0}^{n}\binom{n}{2a}=2^{n-1}$, this can be written as

$$\frac{n+1}{2}\cdot\left(2^{2n-1}+2^{2n-1}\right)+\frac{n+1}{2}\cdot\left(2^{n-1}(2^n-1)+2^{n-1}(2^n-1)\right)$$
$$-(2^n-1)^2+2\left(\frac{n+1}{2}2^{n-1}-\frac{n(n+1)}{2}\right)+2\binom{n+1}{2}+1.$$

We do one last set of combinations of like terms to get $n2^{2n}+2^{n+1}$.

Finally, to get the expected value, we divide by 2^{2n+1}, the number of total ways that we can choose the pieces. This gives our expected value of $n/2+\frac{1}{2^n}$. Finally, plugging in our value of n gives $505+\frac{1}{2^{1010}}$, yielding our answer of 1515.

6. In the country of Princetonia, there are an infinite number of cities, connected by roads. For every two distinct cities, there is a unique sequence of roads that leads from one city to the other. Moreover, there are exactly three roads from every city. On a sunny morning in early July, n tourists have arrived at the capital of Princetonia. They repeat the following process every day: in every city that contains three or more tourists, three tourists are picked and one moves to each of the three cities connected to the original one by roads. If there are 2 or fewer tourists in the city, they do nothing. After some time, all tourists will settle and there will be no more changing cities. For how many values of n from 1 to 2020 will the tourists end in a configuration in which no two of them are in the same city?

   Proposed by Aleksa Milojevic

   Answer: $\boxed{19}$.

   Solution: (*By Daniel Carter*) From the theory of abelian sandpiles, it doesn't matter in what order the cities are considered for relocating tourists (or "collapsed"). Because of this, each successive final configuration may be found by adding one tourist to the capital and settling everything. Denote by $c_n=(a_0,a_1,a_2,...)$ the configuration associated with n tourists, where $a_i\in\{0,1,2\}$ is the number of tourists in any city i away from the capital. By symmetry, all of these cities will have the same number of tourists. Inductively, $c_{3\cdot2^k-4}=(2,2,...,2,0,...)$, $c_{3\cdot2^k-3}=(0,1,1,...,1,0,...)$, and $c_{3\cdot2^k-2}=(1,1,1,...,1,0,...)$, with k twos, k ones, and $k+1$ ones in a row, respectively. This is easily verified for the base case $k=1$, then by the independence of order $c_{2(3\cdot2^k-2)}=c_{3\cdot2^{k+1}-4}=(2,2,...,2,0,...)$ with $k+2$ twos. Adding one more and collapsing the first $k+1$ cities gives $(0,1,1,...,1,3,0,1,0,...)$, $(3,0,1,...,1,0,...)$, $(0,1,...,1,0,...)$ with $k+1$ ones. Adding one more completes the inductive step. Finally, note that for any number strictly between $3\cdot2^k-2$ and $3\cdot2^{k+1}-3$, there is nobody in any city more than k away from the capital, so some city must have two people by Pigeonhole Principle (there's only $3\cdot2^k-2$ cities up to that distance, yet more people). Hence, the condition is met only when $n=3\cdot2^k-2$ or $n=3\cdot2^k-3$ for $k\in\mathbb{N}$, giving 19 solutions (1,3,4,9,10,21,22,...,1534).

7. Let f be defined as below for integers $n \geq 0$ and $a_0, a_1, \ldots$ such that $\sum_{i \geq 0} a_i$ is finite:

$$f(n; a_0, a_1, \ldots) = \begin{cases} a_{2020} & n = 0 \\ \frac{\sum_{i \geq 0} a_i f(n-1; a_0, \ldots, a_{i-1}, a_i - 1, a_{i+1}+1, a_{i+2}, \ldots)}{\sum_{i \geq 0} a_i} & n > 0 \end{cases}.$$

Find the nearest integer to $f(2020^2; 2020, 0, 0, \ldots)$.

Proposed by Daniel Carter

Answer: $\boxed{18}$.

Solution: Consider a balls each uniformly placed into b bins. The value of $f(a; b, 0, 0, \ldots)$ is the expected number of bins containing exactly 2020 balls. In general, the value of $f(n; a_0, a_1, \ldots)$ is the expected number of bins containing exactly 2020 balls given that there are n balls left to place, a_0 bins with no balls, a_1 bins with exactly 1 ball, and so on: if there are no balls left to place the expected number of bins with 2020 balls is a_{2020}, otherwise we place one ball and it has an $a_k / \sum_{i \geq 0} a_i$ chance of being placed into a bin that currently has k balls and the second case of f computes the sum of this over all k.

By linearity of expectation, the expected number of bins containing 2020 balls when there are 2020 bins and 2020^2 total balls is equal to the sum of the probability that any particular bin has 2020 balls over all bins. The chance that any particular bin has 2020 balls is equal to $\binom{2020^2}{2020}\left(\frac{1}{2020}\right)^{2020}\left(\frac{2020-1}{2020}\right)^{2020^2-2020}$, so the desired value of f is 2020 times this.

Now using Stirling's approximation of the factorial, $n! \approx \sqrt{2\pi n}(n/e)^n$, this is very close to

$$x\left(\frac{\sqrt{2\pi x^2}\left(\frac{x^2}{e}\right)^{x^2}}{\sqrt{2\pi x}\left(\frac{x}{e}\right)^x \sqrt{2\pi\left(x^2-x\right)}\left(\frac{x^2-x}{e}\right)^{x^2-x}}\right)\left(\frac{1}{x}\right)^x\left(\frac{x-1}{x}\right)^{x^2-x}$$

where $x = 2020$. This valiantly simplifies to just $\frac{2020}{\sqrt{2\pi}\sqrt{2019}}$, which is very close to $\sqrt{1010/\pi}$. You can use your favorite approximation of π (in fact $\pi \approx 3$ is good enough) to find $1010/\pi \approx 321.5$ is very close to $324 = 18^2$, so the answer is 18.

8. Let $f(k)$ denote the number of triples (a, b, c) of positive integers satisfying $a + b + c = 2020$ with $(k-1)$ not dividing a, k not dividing b, and $(k+1)$ not dividing c. Find the product of all integers k in the range $3 \leq k \leq 20$ such that $(k+1)$ divides $f(k)$.

Proposed by Sunay Joshi

Answer: $\boxed{360}$.

Solution: Let $m = 2020$, for convenience. We use generating functions.

The generating function for a is

$$\frac{1}{1-x} - \frac{1}{1-x^{k-1}} = \frac{x - x^{k-1}}{(1-x)(1-x^{k-1})} = \frac{x(1-x^{k-2})}{(1-x)(1-x^{k-1})}.$$

Similarly, the generating functions for b and c are $\frac{x(1-x^{k-1})}{(1-x)(1-x^k)}$ and $\frac{x(1-x^k)}{(1-x)(1-x^{k+1})}$, respectively. Thus, the generating function for $a + b + c$ is $\frac{x^3 - x^{k+1}}{(1-x)^3(1-x^{k+1})}$.

We must find the coefficient of x^m in this generating function. By adding and subtracting 1 to the numerator, we can rewrite this as

$$\frac{(x^3-1)+(1-x^{k+1})}{(1-x)^3(1-x^{k+1})} = \frac{-(x^2+x+1)}{(1-x)^2(1-x^{k+1})} + \frac{1}{(1-x)^3}.$$

The coefficient of x^m in $\frac{1}{(1-x)^3}$ is $\binom{m+2}{2}$. To find the coefficient of x^m in $\frac{-(x^2+x+1)}{(1-x)^2(1-x^{k+1})}$, we note that the coefficient of x^n in $\frac{1}{(1-x)^2(1-x^{k+1})}$ is the coefficient of x^n in the expansion

$$(1+2x+3x^2+\ldots)(1+x^{k+1}+x^{2(k+1)}+\ldots),$$

i.e.

$$\sum_{t=0}^{t(n,k)}(n+1-t(k+1)),$$

where $t(n,k)=\lfloor\frac{n+1}{k+1}\rfloor$. Expanding this out, we find that the coefficient of x^n is

$$c_n=(n+1)(t(n,k)+1)-\frac{t(n,k)(t(n,k)+1)}{2}(k+1).$$

Since we are looking for the coefficient of x^m in $\frac{-(x^2+x+1)}{(1-x)^2(1-x^{k+1})}$, we want $-(c_m+c_{m-1}+c_{m-2})$. Putting this all together, the coefficient of x^m (i.e. $f(k)$) in our original generating function is given as

$$\begin{aligned}f(k)&=\\ &\binom{m+2}{2}-((m+1)(t(m,k)+1)-\frac{t(m,k)(t(m,k)+1)}{2}(k+1))\\ &-((m)(t(m-1,k)+1)-\frac{t(m-1,k)(t(m-1,k)+1)}{2}(k+1))\\ &-((m-1)(t(m-2,k)+1)-\frac{t(m-2,k)(t(m-2,k)+1)}{2}(k+1)).\end{aligned}$$

Recall that we are only looking at $f(k) \pmod{k+1}$. Since $t(\cdot,\cdot)$ is always an integer, the second terms in the parentheses are divisible by $(k+1)$, and we find

$$\begin{aligned}f(k)&\equiv\binom{m+2}{2}-(m+1)(t(m,k)+1)-(m)(t(m-1,k)+1)-(m-1)(t(m-2,k)+1)\\ &\equiv\binom{m-1}{2}-((m+1)t(m,k)+(m)t(m-1,k)+(m-1)t(m-2,k))\pmod{k+1}.\end{aligned}$$

It remains to check which $k\in[3,20]$ make the right hand side equal 0, i.e. when

$$\binom{m-1}{2}-\left((m+1)\left\lfloor\frac{m+1}{k+1}\right\rfloor+(m)\left\lfloor\frac{m}{k+1}\right\rfloor+(m-1)\left\lfloor\frac{m-1}{k+1}\right\rfloor\right)\equiv 0\pmod{k+1}.$$

A quick computation shows the only possible k are 4, 9, and 10, hence the answer is 360.