Combinatorics A

1. Joey is playing with a 2-by-2-by-2 Rubik's cube made up of 8 1-by-1-by-1 cubes (with two of these smaller cubes along each of the sides of the bigger cubes). Each face of the Rubik's cube is distinct color. However, one day, Joey accidentally breaks the cube! He decides to put the cube back together into its solved state, placing each of the pieces one by one. However, due to the nature of the cube, he is only able to put in a cube if it is adjacent to a cube he already placed. If different orderings of the ways he chooses the cubes are considered distinct, determine the number of ways he can reassemble the cube.

2. Cary has six distinct coins in a jar. Occasionally, he takes out three of the coins and adds a dot to each of them. Determine the number of orders in which Cary can choose the coins so that, eventually, for each number $i \in \{0, 1, \ldots, 5\}$, some coin has exactly i dots on it.

3. Katie has a chocolate bar that is a 5-by-5 grid of square pieces, but she only wants to eat the center piece. To get to it, she performs the following operations:

   i. Take a gridline on the chocolate bar, and split the bar along the line.

   ii. Remove the piece that doesn't contain the center.

   iii. With the remaining bar, repeat steps 1 and 2.

   Determine the number of ways that Katie can perform this sequence of operations so that eventually she ends up with just the center piece.

4. Let $\mathcal{P}$ be the power set of $\{1, 2, 3, 4\}$ (meaning the elements of $\mathcal{P}$ are the subsets of $\{1, 2, 3, 4\}$). How many subsets S of $\mathcal{P}$ are there such that no two distinct integers $a, b \in \{1, 2, 3, 4\}$ appear together in exactly one element of S?

5. Jacob has a piece of bread shaped like a figure 8, marked into sections and all initially connected as one piece of bread. The central part of the "8" is a single section, and each of the two loops of "8" is divided into an additional 1010 pieces. For each section, there is a 50 percent chance that Jacob will decide to cut it out and give it to a friend, and this is done independently for each section. The remaining sections of bread form some number of connected pieces. If E is the expected number of these pieces, and k is the smallest positive integer so that $2^k(E - \lfloor E \rfloor) \geq 1$, find $\lfloor E \rfloor + k$. (Here, we say that if Jacob donates all pieces, there are 0 pieces left).

6. In the country of Princetonia, there are an infinite number of cities, connected by roads. For every two distinct cities, there is a unique sequence of roads that leads from one city to the other. Moreover, there are exactly three roads from every city. On a sunny morning in early July, n tourists have arrived at the capital of Princetonia. They repeat the following process every day: in every city that contains three or more tourists, three tourists are picked and one moves to each of the three cities connected to the original one by roads. If there are 2 or fewer tourists in the city, they do nothing. After some time, all tourists will settle and there will be no more changing cities. For how many values of n from 1 to 2020 will the tourists end in a configuration in which no two of them are in the same city?

7. Let f be defined as below for integers $n \geq 0$ and $a_0, a_1, \ldots$ such that $\sum_{i \geq 0} a_i$ is finite:

$$f(n; a_0, a_1, \ldots) = \begin{cases} a_{2020} & n = 0 \\ \frac{\sum_{i \geq 0} a_i f(n-1; a_0, \ldots, a_{i-1}, a_i - 1, a_{i+1} + 1, a_{i+2}, \ldots)}{\sum_{i \geq 0} a_i} & n > 0 \end{cases}.$$

   Find the nearest integer to $f(2020^2; 2020, 0, 0, \ldots)$.

8. Let $f(k)$ denote the number of triples (a, b, c) of positive integers satisfying $a + b + c = 2020$ with $(k - 1)$ not dividing a, k not dividing b, and $(k + 1)$ not dividing c. Find the product of all integers k in the range $3 \leq k \leq 20$ such that $(k + 1)$ divides $f(k)$.