P U M .·. C

Individual Finals B

1. Find all pairs of natural numbers (n, k) with the following property:

   Given a $k \times k$ array of cells, such that every cell contains one integer, there always exists a path from the left to the right edges such that the sum of the numbers on the path is a multiple of n.

   Note: A path from the left to the right edge is a sequence of cells of the array $a_1, a_2, \ldots, a_m$ so that a_1 is a cell of the leftmost column, a_m is the cell of the rightmost column, and a_i, a_{i+1} share an edge for all $i = 1, 2, \ldots, m-1$.

2. Prove that there is a positive integer M for which the following statement holds:

   For all prime numbers p, there is an integer n for which $\sqrt{p} \leq n \leq M\sqrt{p}$ and $p \mod n \leq \frac{n}{2020}$.

   Note: Here, $p \mod n$ denotes the unique integer $r \in \{0, 1, \ldots, n-1\}$ for which $n | p - r$. In other words, $p \mod n$ is the residue of p upon division by n.

3. Let ABC be a triangle and let the points D, E be on the rays AB, AC such that $BCED$ is cyclic. Prove that the following two statements are equivalent:

   - There is a point X on the circumcircle of ABC such that BDX, CEX are tangent to each other.
   - $AB \cdot AD \leq 4R^2$, where R is the radius of the circumcircle of ABC.