P U M ⁖ C

# Individual Finals A

1. Let $a_1, \ldots, a_{2020}$ be a sequence of real numbers such that $a_1 = 2^{-2019}$, and $a_{n-1}^2 a_n = a_n - a_{n-1}$. Prove that $a_{2020} < \dfrac{1}{2^{2019} - 1}$.

2. Helen has a wooden rectangle of unknown dimensions, a straightedge, and a pencil (no compass). Is it possible for her to construct a line segment on the rectangle connecting the midpoints of two opposite sides, where she cannot draw any lines or points outside the rectangle?

   *Note:* Helen is allowed to draw lines between two points she has already marked, and mark the intersection of any two lines she has already drawn, if the intersection lies on the rectangle. Further, Helen is allowed to mark arbitrary points either on the rectangle or on a segment she has previously drawn. Assume that only the four vertices of the rectangle have been marked prior to the beginning of this process.

3. Let $n$ be a positive integer, and let $\mathcal{F}$ be a family of subsets of $\{1, 2, \cdots, 2^n\}$ such that for any non-empty $A \in \mathcal{F}$ there exists $B \in \mathcal{F}$ so that $|A| = |B| + 1$ and $B \subset A$. Suppose that $\mathcal{F}$ contains all $(2^n - 1)$-element subsets of $\{1, 2, \cdots, 2^n\}$. Determine the minimal possible value of $|\mathcal{F}|$.