# Geometry A Solutions

1. Let $\gamma_1$ and $\gamma_2$ be circles centered at $O$ and $P$ respectively, and externally tangent to each other at point $Q$. Draw point $D$ on $\gamma_1$ and point $E$ on $\gamma_2$ such that line $DE$ is tangent to both circles. If the length $OQ = 1$ and the area of the quadrilateral $ODEP$ is 520, then what is the value of length $PQ$?

   *Proposed by: Ollie Thakar*

   **Answer:** $\boxed{64}$

   Let $r$ be the radius $OQ$ of $\gamma_1$ and $s$ the radius $PQ$ of $\gamma_2$.

   It is a well-known theorem that angle $\angle EQD$ is right, and the length of the hypotenuse $ED$ is $2\sqrt{rs}$.

   Call $a = EQ$ and $b = DQ$. Call $x$ the measure of angle $\angle DQO$. Then, $\angle EQP$ has measure $90 - x$. Furthermore, triangles $DOQ$ and $EPQ$ are isosceles, so $a = 2s \sin x$ and $b = 2s \cos x$.

   Since triangle $EQD$ is right, we have $a^2 + b^2 = ED^2$, which gives us $\sin^2 x = \frac{r}{r+s}$ and $\cos^2 x = \frac{s}{r+s}$.

   The area $A$ of quadrilateral $ODEP$ is given by the sum of the areas of triangles $DOQ$, $EPQ$, and $EDQ$, so:

   $$A = \frac{1}{2}(s\cos x)(2s\sin x)+\frac{1}{2}(r\sin x)(2r\cos x)+\frac{1}{2}4rs\sin x\cos x = (r+s)^2\sin x\cos x = (r+s)\sqrt{rs}.$$

   We are given that $r = 1$ and then that $(1 + s)\sqrt{s} = 520$, which can be solved as $s = 64$ by inspection.

2. Hexagon $ABCDEF$ has an inscribed circle $\Omega$ that is tangent to each of its sides. If $AB = 12$, $\angle FAB = 120^\circ$, and $\angle ABC = 150^\circ$, and if the radius of $\Omega$ can be written as $m + \sqrt{n}$ for positive integers $m, n$, find $m + n$.

   *Proposed by: Sunay Joshi*

   **Answer:** $\boxed{36}$

   Let $r$ denote the radius of $\Omega$, let $O$ denote the center of $\Omega$, and let $\Omega$ touch side $AB$ at point $X$. Then $OX$ is the altitude from $O$ in $\triangle AOB$. Note that $\angle OAB = \frac{1}{2}\angle FAB = 60^\circ$ and $\angle OBA = \frac{1}{2}\angle ABC = 75^\circ$. Thus by right angle trigonometry, $AX = \frac{r}{\tan 60^\circ} = \frac{\sqrt{3}}{3}r$ and $BX = \frac{r}{\tan 75^\circ} = (2-\sqrt{3})r$. As $AB = AX + BX = 12$, we have $(\frac{\sqrt{3}}{3} + 2 - \sqrt{3})r = 12 \to r = 9 + \sqrt{27}$, thus our answer is $m + n = 36$.

3. Let $ABCD$ be a cyclic quadrilateral with circumcenter $O$ and radius 10. Let sides $AB, BC, CD$, and $DA$ have midpoints $M, N, P$, and $Q$, respectively. If $MP = NQ$ and $OM + OP = 16$, then what is the area of triangle $\triangle OAB$?

   *Proposed by: Ollie Thakar*

   **Answer:** $\boxed{78}$

   *Note: The configuration provided in this problem turned out to be impossible, since we arrive at the condition $OM^2+OP^2 = 100$, which cannot hold with the given condition that $OM+OP = 16$. As such, this problem was thrown out during the competition.*

   The condition that $MP = NQ$ is equivalent to the condition that $AC \perp BD$. (This can be seen because the quadrilateral $MNPQ$ is a parallelogram whose sides are parallel to the diagonals $AC$ and $BD$. The condition $MP = NQ$ implies that the parallelogram has equal diagonals, so is a rectangle.) Let $r$ be the circumradius of $ABCD$. By two well-known properties of cyclic orthodiagonal quadrilaterals, we get: $r^2 = AM^2 + CP^2$, $OP = AM$, and $OM = CP$.

Then, Area($\triangle OAB$) $= \frac{1}{2}OM \cdot AB = OM \cdot OP$, and $r^2 = OP^2 + OM^2$.

Thus,

$$\text{Area}(\triangle OAB) = OM \cdot OP = \frac{1}{2}\left((OP+OM)^2 - (OP^2+OM^2)\right) = \frac{1}{2}\left((OP+OM)^2 - r^2\right) = \frac{1}{2}(16^2 - 10^2) = 78.$$

4. Let $C$ be a circle centered at point $O$, and let $P$ be a point in the interior of $C$. Let $Q$ be a point on the circumference of $C$ such that $PQ \perp OP$, and let $D$ be the circle with diameter $PQ$. Consider a circle tangent to $C$ whose circumference passes through point $P$. Let the curve $\Gamma$ be the locus of the centers of all such circles. If the area enclosed by $\Gamma$ is $1/100$ the area of $C$, then what is the ratio of the area of $C$ to the area of $D$?

*Proposed by: Ollie Thakar*

**Answer:** $\boxed{2500}$

Let $r$ be the radius of $C$, and let the length $OP = x$.

First, we prove that $\Gamma$ is an ellipse with foci at $O$ and $P$. Let $X$ be a point on $\Gamma$. Then, draw a circle $E$ centered at $X$ passing through point $P$, tangent to $C$. Since $C$ and $E$ are tangent circles, then $O, X$, and $C$ are collinear. But $XC = XP$, so $r = OC = OX + XC = OX + XP$, so $OX + XP$ is a constant for all $X$ on the curve $\Gamma$, which is the definition of an ellipse.

The area of $\Gamma$ is equal to $\pi$ times the semi-major axis times the semi-minor axis, or, after an application of the Pythagorean theorem: $\pi \cdot \frac{r}{2} \cdot \frac{1}{2}\sqrt{r^2 - x^2}$.

Also by the Pythagorean Theorem, $QP^2 = r^2 - x^2$, so that means the area of $\Gamma$ is $\frac{\pi}{4}rQP$.

By the condition that the area of $C$ is 100 times that of $\Gamma$, then we get that $\pi r^2 = 100\frac{\pi}{4}rQP$, from which we conclude that $\frac{r}{QP} = \frac{100}{4} = 25$, but the ratio of the area of $C$ to the area of $D$ is precisely the square of the ratio $\frac{2r}{QP}$, which is $(2 \cdot 25)^2 = 2500$.

*Note: We initially had the answer of* 625, *but this is incorrect on account of $QP$ being the diameter and not the radius of the circle. We apologize for the confusion this would have caused.*

5. Triangle $ABC$ is so that $AB = 15, BC = 22$, and $AC = 20$. Let $D, E, F$ lie on $BC, AC$, and $AB$, respectively, so $AD, BE, CF$ all contain a point $K$. Let $L$ be the second intersection of the circumcircles of $BFK$ and $CEK$. Suppose that $\frac{AK}{KD} = \frac{11}{7}$, and $BD = 6$. If $KL^2 = \frac{a}{b}$, where $a, b$ are relatively prime integers, find $a + b$.

*Proposed by: Frank Lu*

**Answer:** $\boxed{497}$

First, by Menalaus's theorem, we can compute that $\frac{AK}{KD}\frac{DC}{CB}\frac{BF}{FA} = 1$, which in turn implies that $\frac{BF}{FA} = \frac{7}{11}\frac{22}{16} = \frac{7}{8}$. Therefore, by Ceva's theorem, it follows that $\frac{AE}{EC} = \frac{AF}{FB}\frac{BD}{DC} = \frac{8}{7}\frac{6}{16} = \frac{3}{7}$. From here, we see that $AF = 8, AE = 6$. In particular, notice that by power of point, since $AE \cdot AC = 120 = AB \cdot AF$, it follows that $A$ lies on the radical axes of these circles; in particular, notice that $A, K, L$ are collinear.

Now, notice that the length of $AD$, by Stewart's theorem, is so that $BD \cdot DC \cdot BC + AD^2 \cdot BC = AC^2 \cdot BD + AB^2 \cdot CD$. Plugging in the values we computed, it follows that $6 \cdot 16 \cdot 22 + AD^2 \cdot 22 = 20^2 \cdot 6 + 15^2 \cdot 16 = 3600 + 2400 = 6000$. In particular, it follows that $AD^2 = \frac{6000 - 96 \cdot 22}{22} = \frac{3888}{22} = \frac{1944}{11}$, or that $AD = 18\sqrt{\frac{6}{11}}$. In particular, this means that $AK = \sqrt{66}$. Therefore, computing the power of $A$ again, we see that $AK \cdot AL = 120$ too, meaning that it follows that $AL = \frac{120}{\sqrt{66}} = \frac{20\sqrt{66}}{11}$. Hence, it follows that $KL = \frac{9\sqrt{66}}{11}$, and so that $KL^2 = \frac{486}{11} = 497$.

6. Triangle $ABC$ has side lengths 13, 14, and 15. Let $E$ be the ellipse that encloses the smallest area which passes through $A$, $B$, and $C$. The area of $E$ is of the form $\frac{a\sqrt{b}\pi}{c}$, where $a$ and $c$ are coprime and $b$ has no square factors. Find $a+b+c$.

   *Proposed by: Daniel Carter*

   **Answer:** $\boxed{118}$

   Let $T$ be an affine transformation that sends an equilateral triangle with side length 1 to triangle $ABC$. Affine transformations preserve the ratios of areas, so the smallest such ellipse for the equilateral triangle will be sent to $E$ by $T$. It is clear by inspection that the smallest area ellipse for the equilateral triangle is its circumcircle. The circumcircle of an equilateral triangle has area $\frac{4\sqrt{3}\pi}{9}$ times the area of the triangle, and the area of $ABC$ is 84 (found via Heron's formula), so the area of the $E$ is $\frac{4\sqrt{3}\pi}{9} \cdot 84 = \frac{112\sqrt{3}\pi}{3}$. Thus the answer is $112+3+3=118$.

7. Let $ABC$ be a triangle with sides $AB = 34, BC = 15, AC = 35$ and let $\Gamma$ be the circle of smallest possible radius passing through $A$ tangent to $BC$. Let the second intersections of $\Gamma$ and sides $AB, AC$ be the points $X, Y$. Let the ray $XY$ intersect the circumcircle of the triangle $ABC$ at $Z$. If $AZ = \frac{p}{q}$ for relatively prime integers $p$ and $q$, find $p+q$.

   *Proposed by: Aleksa Milojevic*

   **Answer:** $\boxed{173}$

   There are two solutions: the first one is elementary, but a bit trickier to find, while the second one uses inversion.

   First solution: Let $D$ be the foot of the perpendicular from $A$ to $BC$ (or alternatively the tangency point of $\Gamma$ and $BC$). First, we will extend $XY$ to the other intersection with the circumcircle of $ABC$ and denote that intersection by $W$. Then, we will note that the triangle $AXY$ is similar to $ACB$, by a simple angle chasing argument. Finally, if we denote the tangent to the circumcircle at the point $A$ by $t$, this means that $XY||t$. As $ZW$ is the chord of the circumcircle and it is parallel to $t$, we conclude $AZ = AW$.

   Now, we aim to prove $AZ = AD$. To do this, note the angle equality $\angle ZCA = \angle ZWA = \angle AWZ$, meaning that the triangles $AZY$ and $ACZ$ are similar. This implies $AZ^2 = AY \cdot AC$. Similarly, as $\angle ADC = \angle DYA$, we see that triangles $AYD$ and $ADC$ are also similar, meaning $AD^2 = AY \cdot AC$, which sums up to give $AD = AZ$. From here, we compute $AD$ using Heron's formula: the semiperimeter of the triangle is equal to 42, meaning that the area is $\sqrt{42 \cdot 7 \cdot 8 \cdot 27} = 7 \cdot 9 \cdot 4 = 252$. Therefore, we have that $AZ = AD = \frac{2 \cdot 252}{15} = \frac{168}{5}$, yielding us with $168 + 5 = 173$.

   Second solution. As in the first solution, we aim to show $AD = AZ$ directly. Consider the inversion with the center at $A$ and radius $\sqrt{AY \cdot AC} = \sqrt{AX \cdot AB}$. It swaps pairs of points $(X, B)$ and $(Y, C)$. Thus, the circle $ABC$ is sent to the line $XY$. Note that $Z$, defined as the intersection of $XY$ and the circumcircle must therefore be sent to the intersection of the circumcircle and $XY$, i.e. to $Z$. Thus, $Z$ stays fixed under this inversion. Similarly, $D$ is the intersection of the circumcircle of $AXY$ and the line $BC$. As these two objects are swapped under this inversion, by an argument similar to the one for $Z$, we see that $D$ stays constant under this inversion. But, this means that $AD = \sqrt{AY \cdot AC} = AZ$, which completes the solution.

8. $A_1A_2A_3A_4$ is a cyclic quadrilateral inscribed in circle $\Omega$, with side lengths $A_1A_2 = 28$, $A_2A_3 = 12\sqrt{3}$, $A_3A_4 = 28\sqrt{3}$, and $A_4A_1 = 8$. Let $X$ be the intersection of $A_1A_3, A_2A_4$. Now, for $i = 1, 2, 3, 4$, let $\omega_i$ be the circle tangent to segments $A_iX, A_{i+1}X$, and $\Omega$, where we take indices cyclically $\pmod 4$. Furthermore, for each $i$, say $\omega_i$ is tangent to $A_1A_3$ at $X_i$, $A_2A_4$ at $Y_i$, and $\Omega$ at $T_i$. Let $P_1$ be the intersection of $T_1X_1$ and $T_2X_2$, and $P_3$ the intersection of $T_3X_3$

and $T_4X_4$. Let $P_2$ be the intersection of $T_2Y_2$ and $T_3Y_3$, and $P_4$ the intersection of $T_1Y_1$ and $T_4Y_4$. Find the area of quadrilateral $P_1P_2P_3P_4$.

*Proposed by: Frank Lu*

**Answer:** $\boxed{784}$

First, we claim that the $P_i$ all lie on the circle. To show this, we first claim that $P_1, P_3$ are midpoints of opposite arcs for $A_1A_3$. To see this, we first notice that $P_1$ is the midpoint of the arc $A_1A_3$ opposite of $T_1$. To see this, notice that the midpoint of this arc lies on $T_1X_1$; we can see this by taking a homothety centered at $T_1$, which takes $\omega_1$ to $\Omega$. This is a well-known result of a circle inscribed in a segment. But this holds for $T_2X_2$ as well, meaning that this point is $P_1$. A similar result holds for $P_2, P_3, P_4$; in particular, these all lie on the circle. Notice furthermore that $P_1P_3$ and $P_2P_4$ are diameters, meaning that in particular we see that $P_1P_2P_3P_4$ this is a rectangle.

From here, we need to find the side lengths of the rectangle, which requires us to first find the circumradius of this quadrilateral. To see this, we first find the diagonal length $A_1A_3$, which we set as $c$, and the angle $\angle A_1A_2A_3$, which we set as $\alpha$. Then, we know that $c^2 = 784+432-2\cdot 28\cdot 12\sqrt{3}\cos\alpha$. But by cyclic quadrilaterals, we also know that $c^2 = 2352+64+2\cdot 28\sqrt{3}\cdot 8\cos\alpha$. Equating these two, we find that $2\cdot 28\sqrt{3}\cdot 20 = -2416+1216 = 1200$, or that $\cos\alpha = \frac{-5\sqrt{3}}{14}$. From here, we see that $\sin\alpha = \frac{11}{14}$. Therefore, we see that $A_1A_3^2 = 784+432+48\sqrt{3}\cdot 5\sqrt{3} = 1216+720 = 1936$, and so thus $A_1A_3 = 44$. It thus follows that the circumradius is $R = \frac{44}{\frac{22}{14}} = 28$.

Finally, we need to find the angle $\theta$ between $A_1A_3$ and $A_2A_4$, as then the angle between $P_1P_3$ and $P_2P_4$ is equal to $\theta$ as well. We now seek to find this angle.

To find this angle, notice that the area of this quadrilateral is going to be the product of the diagonals times the sine of the angle between these diagonals times $\frac{1}{2}$. But by Ptolemy's, the product of this is $28\cdot 28\sqrt{3}+8\cdot 12\sqrt{3} = (784+96)\sqrt{3} = 880\sqrt{3}$. As for the area of the quadrilateral, we employ the fact that we know the side lengths and split this up into four triangles: we see that the corresponding inscribed angles to them have sines $\frac{1}{2}$, $\frac{3\sqrt{3}}{14}$, $\frac{\sqrt{3}}{2}$, and $\frac{1}{7}$, respectively. Therefore, we see that the sines of the central angles are equal to $\frac{\sqrt{3}}{2}$, $\frac{39\sqrt{3}}{98}$, $\frac{\sqrt{3}}{2}$, and $\frac{8\sqrt{3}}{49}$. Finally, notice that the sum of the cosines of the three smaller angles yields us the larger angle, meaning that the center doesn't lie in this quadrilateral. To verify this: we see that the sum of the first two angles has cosine of $\frac{\sqrt{3}}{2}\frac{13}{14}-\frac{1}{2}\frac{3\sqrt{3}}{14} = \frac{5\sqrt{3}}{14}$, with sine $\frac{11}{14}$, and adding this to the next smaller angle yields us a cosine of $\frac{5\sqrt{3}}{14}\frac{4\sqrt{3}}{7}-\frac{11}{14}\frac{1}{7} = \frac{1}{2}$.

Our area of the quadrilateral $A_1A_2A_3A_4$ is thus $\frac{1}{2}28^2(\frac{\sqrt{3}}{2}+\frac{39\sqrt{3}}{98}-\frac{\sqrt{3}}{2}+\frac{8\sqrt{3}}{49}) = 156\sqrt{3}+64\sqrt{3} = 220\sqrt{3}$. But this means that the angle between the two diagonals has a sine of $\frac{1}{2}$. But this means our rectangle has area of $\frac{1}{2}56^2\frac{1}{2} = 784$.