P U M $\therefore$ C

Algebra B

1. The function $f(x) = x^2 + (2a+3)x + (a^2+1)$ only has real zeroes. Suppose the smallest possible value of a can be written in the form $\frac{p}{q}$, where p, q are relatively prime integers. Find $|p| + |q|$.

2. Princeton has an endowment of 5 million dollars and wants to invest it into improving campus life. The university has three options: it can either invest in improving the dorms, campus parties or dining hall food quality. If they invest a million dollars in the dorms, the students will spend an additional $5a$ hours per week studying. If the university invests b million dollars in better food, the students will spend an additional $3b$ hours per week studying. Finally, if the c million dollars are invested in parties, students will be more relaxed and spend $11c - c^2$ more hours per week studying. The university wants to invest its 5 million dollars so that the students get as many additional hours of studying as possible. What is the maximal amount that students get to study?

3. Let $f(x) = \frac{x+a}{x+b}$ satisfy $f(f(f(x))) = x$ for real numbers a, b. If the maximum value of a is $\frac{p}{q}$, where p, q are relatively prime integers, what is $|p| + |q|$?

4. Let C denote the curve $y^2 = \frac{x(x+1)(2x+1)}{6}$. The points $(\frac{1}{2}, a), (b, c)$, and $(24, d)$ lie on C and are collinear, and $ad < 0$. Given that b, c are rational numbers, find $100b^2 + c^2$.

5. Let $\{x\} = x - \lfloor x \rfloor$. Consider a function f from the set $\{1, 2, \ldots, 2020\}$ to the half-open interval $[0, 1)$. Suppose that for all x, y, there exists a z so that $\{f(x) + f(y)\} = f(z)$. We say that a pair of integers m, n is valid if $1 \leq m, n \leq 2020$ and there exists a function f satisfying the above so $f(1) = \frac{m}{n}$. Determine the sum over all valid pairs m, n of $\frac{m}{n}$.

6. Let P be a 10-degree monic polynomial with roots $r_1, r_2, \ldots, r_{10} \neq 0$ and let Q be a 45-degree monic polynomial with roots $\frac{1}{r_i} + \frac{1}{r_j} - \frac{1}{r_i r_j}$ where $i < j$ and $i, j \in \{1, \ldots, 10\}$. If $P(0) = Q(1) = 2$, then $\log_2(|P(1)|)$ can be written as $\frac{a}{b}$ for relatively prime integers a, b. Find $a + b$.

7. Suppose we have a sequence $a_1, a_2, \ldots$ of positive real numbers so that for each positive integer n, we have that $\sum_{k=1}^{n} a_k a_{\lfloor \sqrt{k} \rfloor} = n^2$. Determine the first value of k so $a_k > 100$.

8. Given integer n, let W_n be the set of complex numbers of the form $re^{2qi\pi}$, where q is a rational number so that $qn \in \mathbb{Z}$ and r is a real number. Suppose that p is a polynomial of degree ≥ 2 such that there exists a non-constant function $f : W_n \to \mathbb{C}$ so that $p(f(x))p(f(y)) = f(xy)$ for all $x, y \in W_n$. If p is the unique monic polynomial of lowest degree for which such an f exists for $n = 65$, find $p(10)$.