Algebra A Solutions

1. Let $f(x)=\frac{x+a}{x+b}$ satisfy $f(f(f(x)))=x$ for real numbers a, b. If the maximum value of a is $\frac{p}{q}$, where p,q are relatively prime integers, what is $|p|+|q|$?

   Proposed by: Henry Erdman

   Answer: $\boxed{7}$

   Substituting in $f(x)$ for x in $f(x)$ twice yields that $f(f(f(x)))=\frac{(1+2a+ab)x+(a+a^2+ab+ab^2)}{(1+a+b+b^2)x+(a+2ab+b^3)}$. We note that the coefficient of x in the denominator must be zero, and thus we have that $a=-b^2-b-1$. This parabola opens down and has its vertex at $b=-\frac{1}{2}$, giving an upper limit on a of $-\frac{3}{4}$. We now need to verify that $(a,b)=(-\frac{3}{4},-\frac{1}{2})$ satisfies the rest of the problem. We have $1+2a+ab=1-\frac{3}{2}+\frac{3}{8}=-\frac{1}{8}$ as the coefficient of x in the numerator, $-\frac{3}{4}+\frac{9}{16}+\frac{3}{8}-\frac{3}{16}=0$ as the constant in the numerator, and $-\frac{3}{4}+\frac{3}{4}-\frac{1}{8}=-\frac{1}{8}$ as the constant in the denominator. Thus, we do indeed have a solution, and it is the greatest possible value of a. So, our answer is $|-3|+|4|=\boxed{7}$.

2. Let C denote the curve $y^2=\frac{x(x+1)(2x+1)}{6}$. The points $(\frac{1}{2},a),(b,c)$, and $(24,d)$ lie on C and are collinear, and $ad<0$. Given that b,c are rational numbers, find $100b^2+c^2$.

   Proposed by: Sunay Joshi

   Answer: $\boxed{101}$

   By plugging $x=\frac{1}{2}$ into the equation for C, we find $a=\mp\frac{1}{2}$. Similarly, $d=\pm 70$. Since $ad<0$, there are only two possible pairs (a,d), namely $(a,d)=(-\frac{1}{2},70)$ or $(\frac{1}{2},-70)$.

   Suppose $(a,d)=(-\frac{1}{2},70)$. Then the equation of the line through $(\frac{1}{2},-\frac{1}{2})$ and $(24,70)$ is $y=3x-2$. Plugging this into the equation for C, we find $(3x-2)^2=\frac{x(x+1)(2x+1)}{6}$. Simplifying, we find $2x^3-51x^2+\ldots=0$.

   At this point, instead of solving this equation explicitly, we use a trick. Since $(\frac{1}{2},-\frac{1}{2})$ and $(24,-70)$ lie on this line, $x=\frac{1}{2}$ and $x=24$ are roots of this cubic. Thus, the remaining root $x=b$ must satisfy Vieta's Formula for the sum of roots! We get $b+\frac{1}{2}+24=\frac{51}{2}$, thus $b=1$. Plugging this into the equation of our line, we find $c=1$, hence $(b,c)=(1,1)$.

   By the symmetry of C across the x axis, the other case yields $(b,c)=(1,-1)$. In either case, we find an answer of $100\cdot 1^2+1^2=\boxed{101}$.

3. Let $\{x\}=x-\lfloor x\rfloor$. Consider a function f from the set $\{1,2,\ldots,2020\}$ to the half-open interval $[0,1)$. Suppose that for all x,y, there exists a z so that $\{f(x)+f(y)\}=f(z)$. We say that a pair of integers m,n is valid if $1\le m,n,\le 2020$ and there exists a function f satisfying the above so $f(1)=\frac{m}{n}$. Determine the sum over all valid pairs m,n of $\frac{m}{n}$

   Proposed by: Frank Lu

   Answer: $\boxed{1019595}$

   We will consider the set of all possible images for f, as this is the only restriction we are given on our function.

   First, suppose that $f(x)$ was irrational for some value of x. Then, it follows that $\{n*f(x)\}$ is in the image of f for all $n\in\mathbb{N}$. But this is impossible since our domain has only finitely many elements. Thus, it follows that our function can only be rational-valued. By repeating this argument, we also know that the denominator of $f(x)$ must be at most 2020.

   We now claim that all such values are valid for $f(1)$. To see this, let $f(x)=\{xf(1)\}$. The fact that our condition is satisfied is clear. We thus find $\sum_{i=1}^{2020}\sum_{j=0}^{i-1}\frac{j}{i}=\sum_{i=1}^{2020}\frac{i-1}{2}=2019*505=$ $\boxed{1019595}$ is our answer.

4. Let P be a 10-degree monic polynomial with roots $r_1, r_2, \dots, r_{10} \neq 0$ and let Q be a 45-degree monic polynomial with roots $\frac{1}{r_i} + \frac{1}{r_j} - \frac{1}{r_i r_j}$ where $i < j$ and $i, j \in \{1, \dots, 10\}$. If $P(0) = Q(1) = 2$, then $\log_2(|P(1)|)$ can be written as $\frac{a}{b}$ for relatively prime integers a, b. Find $a + b$.

Proposed by: Matthew Kendall

Answer: $\boxed{19}$

We can factor Q as a product of its roots:

$$Q(x) = \prod_{i<j} \left(x - \frac{1}{r_i} - \frac{1}{r_j} + \frac{1}{r_i r_j} \right).$$

Then we see

$$Q(1) = \prod_{i<j} \left(1 - \frac{1}{r_i} - \frac{1}{r_j} + \frac{1}{r_i r_j} \right) = \prod_{i<j} \frac{1}{r_i r_j}(1 - r_i)(1 - r_j) = \frac{1}{(r_1 r_2 \cdots r_{10})^9} P(1)^9.$$

Hence $\frac{1}{2^9}|P(1)|^9 = 2$, so $|P(1)| = 2^{\frac{10}{9}}$, giving an answer of $\boxed{19}$.

5. Suppose we have a sequence $a_1, a_2, \dots$ of positive real numbers so that for each positive integer n, we have that $\sum_{k=1}^{n} a_k a_{\lfloor \sqrt{k} \rfloor} = n^2$. Determine the first value of k so $a_k > 100$.

Proposed by: Frank Lu

Answer: $\boxed{1018}$

Note: On the original algebra test, we had forgotten to include the phrase "positive real numbers."

Notice that this relation becomes the equation that $a_n = \frac{2n-1}{a_{\lfloor \sqrt{n} \rfloor}}$, by subtracting this for n and $n - 1$. From here, to figure out when this is larger than 100, we need to make some deductions about the rough behavior of this sequence. Notice here that, trying smaller values, we have that $a_2 = 3, a_3 = 5, a_4 = 7/3, a_5 = 3, a_6 = 11/3$.

First, notice that $a_n = \frac{2n-1}{2\lfloor \sqrt{n} \rfloor - 1} a_{\lfloor \sqrt{\lfloor \sqrt{n} \rfloor} \rfloor} > \sqrt{n} a_{\lfloor \sqrt{\lfloor \sqrt{n} \rfloor} \rfloor}$. Observe then that for $n = 1295 = 6^4 - 1$, notice that $a_{1295} > 35 a_5 = 105$, so hence our maximal value is going to be at most 1295. In particular, we see that $a_{\lfloor \sqrt{\lfloor \sqrt{n} \rfloor} \rfloor}$ for our maximal value is either going to be a_1, a_2, a_3, a_4, or a_5. But notice however that $\lfloor \sqrt{\lfloor \sqrt{n} \rfloor} \rfloor = 5$; if it is 4, notice that this is at most $\frac{7}{3} \frac{2n-1}{2\lfloor \sqrt{n} \rfloor - 1} < \frac{7}{3} \frac{1250}{31} = \frac{8750}{93} < 100$. And furthermore, if it is less than 4, we see that we can bound this more crudely by $5 \frac{2n-1}{2\lfloor \sqrt{n} \rfloor - 1} < 5 \frac{2(\lfloor \sqrt{n} \rfloor + 1)^2 - 1}{2\lfloor \sqrt{n} \rfloor - 1} = 5 \frac{2\lfloor \sqrt{n} \rfloor^2 + 4\lfloor \sqrt{n} \rfloor + 1}{2\lfloor \sqrt{n} \rfloor - 1} = 5(\lfloor \sqrt{n} \rfloor + 5/2 + \frac{7/2}{2\lfloor \sqrt{n} \rfloor - 1})$. On the one hand, we see that if $\lfloor \sqrt{n} \rfloor \leq 3$, this is at most $5(3 + 5/2 + 7/2) < 45$. On the other hand, $\lfloor \sqrt{n} \rfloor \geq 4$, so this is at most $5(15 + 5/2 + 1/2) < 90 < 100$.

In particular, we require then that for our minimal value for n, we have that $a_n = \frac{6n-3}{2\lfloor \sqrt{n} \rfloor - 1}$. On one hand, again we can use our bounds above to see that this is bounded above by $3(\lfloor \sqrt{n} \rfloor + 5/2 + \frac{7/2}{2\lfloor \sqrt{n} \rfloor - 1})$; we therefore need to have that $\lfloor \sqrt{n} \rfloor + 5/2 + \frac{7/2}{2\lfloor \sqrt{n} \rfloor - 1} > 33$. But with $\sqrt{n} \geq 25$ in this particular subcase, this means that we have that $\lfloor \sqrt{n} \rfloor + 3 > 33$, or that $\lfloor \sqrt{n} \rfloor > 30$. We start with this being 31; we then get that $a_n = \frac{6n-3}{61}$. To be larger than 100, this requires that $6n > 6103$, or that $n \geq \boxed{1018}$.

6. Given integer n, let W_n be the set of complex numbers of the form $re^{2qi\pi}$, where q is a rational number so that $qn \in \mathbb{Z}$ and r is a real number. Suppose that p is a polynomial of degree ≥ 2 such that there exists a non-constant function $f : W_n \to \mathbb{C}$ so that $p(f(x))p(f(y)) = f(xy)$ for

all $x, y \in W_n$. If p is the unique monic polynomial of lowest degree for which such an f exists for $n = 65$, find $p(10)$.

Proposed by: Frank Lu

Answer: $\boxed{100009}$

Note: On the original algebra test, we had forgotten the phrase "r is a real number."

Fix $f(1)$ and $p(x)$.

First, note that plugging in $x = y = 1$ yields that $p(f(1))^2 = f(1)$, and $y = 1$ yields that $p(f(x))p(f(1)) = f(x)$.

Hence, we see that the image of f is a root of the polynomial $p(u)p(f(1)) - u = 0$, which in particular means that f has a finite image. Furthermore, we thus see that $p(f(x))p(f(y))p(f(1))^2 = f(xy)p(f(1))^2$, which means that, in fact, that $f(x)f(y) = f(1)f(xy) \forall x, y \in W_n$. If $f(1)$ is zero, then it follows that $\forall x \in W_n$ that $f(x) = 0$, so we consider when $f(1) \neq 0$. Then, we see that, letting $g(x) = f(x)/f(1)$ that $g(x)g(y) = g(xy) \forall x, y \in W_n$.

Since the image of g is finite, if there exists a value of x so that $|g(x)| \neq 1, |g(x)| \neq 0$, then $g(x), g(x^2), \ldots$ are all distinct, contradiction. Furthermore, $g(x) = 0$ for some $x \neq 0$ means that $g(x)g(y/x) = 0 = g(y) \forall y \in W_n$, so we take that we want $|g(x)| = 1$ for all $x \in W_n$. By a similar logic, we see that $g(x)$ must be a root of unity, as again we will run into the issue where the image of g is infinite.

We see that if p is a prime not dividing n, then $g(x)$ can't ever be a pth root of unity, since otherwise we could take the pth root of x to get another root (raising all of the roots to the pth power yields a permutation of the roots). Thus, we see that the minimal possible value for the degree of our polynomial is 5, which would then require it to have the 5th roots of unity as roots.

Thus, we see that $p(u)p(f(1)) - u = au^5 - a$ for some complex number a, meaning that $p(u) = \frac{a}{p(f(1))}u^5 + u - \frac{a}{p(f(1))}$, which we can just write as $p(u) = u + c(u^5 - 1)$ for some complex constant c. By monic, we see that $p(x) = x^5 + x - 1$ yields that $p(10) = \boxed{100009}$.

7. Suppose that p is the unique monic polynomial of minimal degree such that its coefficients are rational numbers and one of its roots is $\sin\frac{2\pi}{7} + \cos\frac{4\pi}{7}$. If $p(1) = \frac{a}{b}$, where a, b are relatively prime integers, find $|a + b|$.

Proposed by: Frank Lu

Answer: $\boxed{57}$

We'll first find the polynomial with roots that are $\sin\frac{2n\pi}{7} + \cos\frac{4n\pi}{7}$, where n goes from 1 to 6 and are integers. Then, we'll show that this has minimal degree. Let this polynomial be q. First, notice that $\prod_{n=1}^{6}(x - \sin\frac{2n\pi}{7} - \cos\frac{4n\pi}{7}) = \prod_{n=1}^{6}(x + 2\sin^2\frac{2n\pi}{7} - \sin\frac{2n\pi}{7} - 1)$.

However, notice also that $q(\frac{-x^2-3x}{2}) = \prod_{n=1}^{6}(\frac{-x^2-3x}{2} + 2\sin^2\frac{2n\pi}{7} - \sin\frac{2n\pi}{7} - 1) = \prod_{n=1}^{6}(x + 2\sin\frac{2n\pi}{7} + 1)(-\frac{x}{2} + \sin\frac{2n\pi}{7} - 1)$. Suppose that h is the monic polynomial with roots being the $\sin\frac{2n\pi}{7}$. Then, notice that this is equal to $64h(-(x+1)/2)h(x/2+1)$.

We can explicitly find what h is, however. Notice that the equation giving that $\sin 7\theta = 0$, using DeMoivre's theorem, yields us the equation $-\sin^7\theta + 21\sin^5\theta\cos^2\theta - 35\sin^3\theta\cos^4\theta + 7\sin\theta\cos^6\theta = 0$, or that $-\sin^7\theta + 21\sin^5\theta(1 - \sin^2\theta) - 35\sin^3\theta(1 - \sin^2\theta)^2 + 7\sin\theta(1 - \sin^2\theta)^3 = 0$.

Expanding this out, we see that this is $-\sin^7\theta + 21\sin^5\theta - 21\sin^7\theta - 35\sin^3\theta + 70\sin^5\theta - 35\sin^7\theta + 7\sin\theta - 21\sin^3\theta + 21\sin^5\theta - 7\sin^7\theta = 0$. Simplifying, this is $-64\sin^7\theta + 112\sin^5\theta -$

$56\sin^3\theta + 7\sin\theta = 0$. Notice that this has 7 roots, but one of these is just 0; this yields us that, in fact, $h(x) = x^6 - \frac{7}{4}x^4 + \frac{7}{8}x^2 - \frac{7}{64}$. Furthermore, we see that this polynomial cannot be factored further in the rationals; we can check this using Eisenstein's criterion, for instance.

From here, we will show that, in fact, $q = p$. Once we have this, we see that $p(1) = q(1) = q(\frac{-(-2)^2-3*(-2)}{2}) = 64h(1/2)h(0) = 64(\frac{1-7+7\cdot 2-7}{64})(\frac{-7}{64}) = -\frac{7}{64}$, which would yield our desired answer of $\boxed{57}$.

To show that p is q, we know that p has to divide q. But in fact, notice that q has to be at least degree 3, since $p(-2x^2+x+1)$ is a polynomial where $\sin\frac{2\pi}{7}$ is a root, so is divisible by a sixth degree polynomial h. But notice that $-2x^2+x+1+2\sin^2\frac{2\pi}{7} - \sin\frac{2\pi}{7} - 1 = (x - \sin\frac{2\pi}{7})(-2x - 2\sin\frac{2\pi}{7} - 1)$. However, notice that none of the other roots of $p(-2x^2+x+1)$ are roots of h; otherwise we have that $-\sin\frac{2n\pi}{7} - \frac{1}{2} = \sin\frac{2m\pi}{7}$ for some integers m, n, or that $-\frac{1}{2} = \sin\frac{2m\pi}{7} + \sin\frac{2n\pi}{7}$. But we see that this doesn't occur; indeed, notice that both of these sines can't be negative (as $\sin\frac{\pi}{7} > \sin\frac{\pi}{12} = \frac{\sqrt{6}-\sqrt{2}}{4} > \frac{1}{4}$), and if one is negative and one is positive, we require that either $\frac{1}{2}$ is $\sin\frac{3\pi}{7} - \sin\frac{\pi}{7}$ or $\sin\frac{2\pi}{7} - \sin\frac{\pi}{7}$.

None of these hold, though, as the second is $2\cos\frac{3\pi}{14}\sin\frac{\pi}{14} < \sqrt{3}\sin\frac{\pi}{12} = \frac{3\sqrt{2}-\sqrt{6}}{2}$, and the other is $2\cos\frac{2\pi}{7}\sin\frac{\pi}{7}$. But if this is $1/2$, this means that $2\sin\frac{\pi}{7} - 4\sin^3\frac{\pi}{7} - 1/2 = 0$, which is not possible as we deduced that the polynomial of minimal degree is degree 6 for $\sin\frac{\pi}{7}$.

This forces us to have $p = q$, as desired.

8. Let a_n be the number of unordered sets of three distinct bijections $f, g, h : \{1, 2, ..., n\} \to \{1, 2, ..., n\}$ such that the composition of any two of the bijections equals the third. What is the largest value in the sequence $a_1, a_2, \ldots$ which is less than 2021?

   Proposed by: Austen Mazenko

   Answer: $\boxed{875}$

   First, $h := f \circ g = g \circ f$, so $f(h(x)) = f(g(f(x))) = g(x)$. Since g is bijective, this holds iff $g(f(g(f(x)))) = h(h(x)) = g(g(x)))$, so by analogous equations we find $f^2 = g^2 = h^2$. But, we also have $h(f(x)) = g(x) \implies g(f(f(x))) = g^3(x) = g(x) \implies g^2(x) \equiv x$; analogous reasoning holds for the other two functions, so they must be involutions.

   Suppose f's cycles are $(a_1, b_1), (a_2, b_2), ..., (a_n, b_n)$ (meaning $f(a_1) = b_1, f(b_1) = a_1)$) while every other value is a fixed point of f. We will consider the number of possibilities for g (each of which fixes h). To start, note $f(g(a_1)) = g(f(a_1)) \implies f(g(a_1)) = g(b_1)$. If $g(a_1) = a_1$, then $g(b_1) = b_1$ so a_1, b_1 are fixed points of g and (a_1, b_1) is a cycle in h. If $g(a_1) = b_1$, then (a_1, b_1) is a cycle in g, and a_1, b_1 are fixed points in h. If $g(a_1) = a_i$ or b_i for some $i > 1$, then $g(b_1) = b_i$, so g has cycles $(a_1, a_i), (b_1, b_i)$. Furthermore, $f(g(a_1)) = b_i$, $(a_1, b_i), (a_i, b_1)$ are cycles in h. Finally, $g(a_1)$ cannot be a fixed point of f since then $f(g(a_1)) = g(a_1) = g(b_1)$, contradicting bijectivity. Analogous reasoning holds for the other cycles of f.

   The other possibility is to let x_1 be a fixed point of f, and consider $f(g(x_1)) = g(f(x_1)) = g(x_1)$; hence, $g(x_1)$ is also a fixed point of f. Either $g(x_1) = x_1$, meaning $g(x_1) = x_1$ and $h(x_1) = x_1$, or $g(x_1) = x_2$ for some x_2, implying $h(x_1) = x_2$.

   Combining the above information is sufficient to form a recursion for a_n. Evidently, $a_0 = a_1 = a_2 = a_3 = 0$. Now, for $n \geq 4$ there are a few possibilities. First, n could be a fixed point of f, g, and h, giving a_{n-1} possibilities. Second, n could be paired with some other value m such that (m, n) is a cycle in two of f, g, h and fixed by the third. There are $n-1$ ways to select m, 3 ways to determine which of f, g, h will fix m and n, and then a_{n-2} triplets to pick from. However, this situation is also possible when two of f, g, h are identical on $\{1, 2, ..., n-1\} \setminus \{m\}$, and the third is the identity function on this set. WLOG $f \equiv g$ and h is the identity: if f fixes m, n while g does not, this will make f, g, h different on $\{1, 2, ..., n\}$. The number of ways for $f \equiv g$ is simply the number of involutions on $n-2$ elements, minus 1 for the case when f, g, h are all the identity bijection. Let b_n denote the number of involutions on n elements. Evidently $b_0 = 1, b_1 = 1$, and for $n \geq 2$ either n is fixed or it's transposed with one of the other $n-1$ terms, so $b_n = b_{n-1} + (n-1)b_{n-2}$. Hence, starting with index 0, the sequence $\{b_n\}$ is $1, 1, 2, 4, 10, 26, 76, ...$ Thus, this situation adds $(n-1)(b_{n-2}-1)$ to our count.

The third and final possibility is that n is part of a cycle which is "paired" with another cycle. This corresponds to the previously outlined scenario when $(a_1, b_1), (a_i, b_i)$ are cycles of f and (a_1, a_i) or (a_1, b_i) is a cycle of g, in which case (a_1, b_i) or (a_1, a_i), respectively, is a cycle of h. If n is in such a pairing, there are $\binom{n-1}{3}$ ways to select the other three values. Then, if f, g, h are distinct when restricted to the set excluding these four values, there are 3! ways to assign the cycles, contributing $6\binom{n-1}{3}a_{n-4}$ cases. As before, if exactly two of f, g, h are the same, we will have 3 ways to assign the cycles, so this case contributes $3 \cdot \binom{n-1}{3}(b_{n-4}-1)$ to our tally. Finally, if f, g, h are each the identity on the restriction to all but the four values of interest, we get an additional $\binom{n-1}{3}$ possibilities.

Hence,

$$a_n = a_{n-1} + 3(n-1) \cdot a_{n-2} + (n-1) \cdot (b_{n-2}-1) + 6 \cdot \binom{n-1}{3} \cdot a_{n-4} + 3 \cdot \binom{n-1}{3} \cdot (b_{n-4}-1) + \binom{n-1}{3}.$$

Simply plugging into the recurrence gives $a_4 = 4, a_5 = 20, a_6 = 165$, and $a_7 = 875$. It is evident a_8 is too large and the sequence is monotonically increasing, so our answer is $\boxed{875}$.