Number Theory B

1. The product of the positive factors of a positive integer n is 8000. What is n?

2. The least common multiple of two positive integers a and b is $2^5 \times 3^5$. How many such ordered pairs (a, b) are there?

3. Let f be a function over the natural numbers so that

   (a) $f(1) = 1$

   (b) If $n = p_1^{e_1} ... p_k^{e_k}$ where $p_1, \cdots, p_k$ are distinct primes, and $e_1, \cdots e_k$ are non-negative integers, then $f(n) = (-1)^{e_1 + .. + e_k}$.

   Find $\sum_{i=1}^{2019} \sum_{d|i} f(d)$.

4. Let n be the smallest positive integer which can be expressed as a sum of multiple (at least two) consecutive integers in precisely 2019 ways. Then n is the product of k not necessarily distinct primes. Find k.

5. Consider the first set of 38 consecutive positive integers who all have sum of their digits not divisible by 11. Find the smallest integer in this set.

6. Let f be a polynomial with integer coefficients of degree 2019 such that the following conditions are satisfied:

   (a) For all integers n, $f(n) + f(-n) = 2$.

   (b) $101^2 \mid f(0) + f(1) + f(2) + \cdots + f(100)$.

   Compute the remainder when $f(101)$ is divided by 101^2.

7. For a positive integer n, let $f(n) = \sum_{i=1}^{n} \lfloor \log_2 i \rfloor$. Find the largest $n < 2018$ such that $n \mid f(n)$.

8. Call a positive integer n *compact* if for any infinite sequence of distinct primes $p_1, p_2, \ldots$ there exists a finite subsequence of n primes $p_{x_1}, p_{x_2}, \ldots p_{x_n}$ (where the x_i are distinct) such that

$$p_{x_1} p_{x_2} \cdots p_{x_n} \equiv 1 \pmod{2019}$$

   Find the sum of all *compact* numbers less than $2 \cdot 2019$.