P U M ∴ C

Individual Finals B

1. Find all nonnegative integers n and m such that $2^n = 7^m + 9$.

   Answer: $n = 4, m = 1$ is the only solution.

   Solution: When we look at the equation $\pmod 3$ we get that n is even, since $7 \equiv 1 \pmod 3$. Then $n = 2k$, where k is a nonnegative integer. Then $(2^k - 3)(2^k + 3) = 7^m$, so both $2^k - 3$ and $2^k + 3$ are powers of 7. If $k \geq 3$, this is impossible by looking at $\pmod 8$ for $2^k - 3$. Then we check $k = 0, 1, 2$ to get that $k = 2$ is a solution. Then $n = 4, m = 1$ is the only solution

   Proposed by Igor Medvedev and Aleksa Milojević.

2. Let $G = (V, E)$ be a connected graph. Show that there exists a subset $F \subseteq E$ such that every vertex in $H = (V, F)$ has odd degree if and only if $|V|$ is even.

   Note: A *connected graph* is a graph such that for any two vertices there is a path from one to the other.

   Solution: Suppose first that $|V|$ is even and proceed by induction. Suppose the contrary, that there is no such F. Then take a spanning tree of G. By the assumption, this spanning tree does not have all the vertices with odd degree, so there exists a vertex v with even degree. Now make v the root of the spanning tree. Then one of the subtrees of v has an odd number of vertices, because the total number of vertices is even. Let the vertex in that subtree which is a child of v be u. Then u has an even number of children, call this set V_1. Let V_2 be the set of all the other children of v except u and V_1. Then apply the inductive hypothesis to the induced graphs on V_1 and V_2, and suppose we get sets F_1 and F_2 of edges. Then the set $F_1 \cup F_2 \cup \{uv\}$ satisfies the desired property.

   Now if the graph G satisfies this property, then we can apply the usual double counting formula for the subgraph of G with edges F to get that $\sum_{v \in V} d_F(v) = 2|F|$, where $d_F(v)$ denotes the degree of v in the graph on V with edges F. Then each $d_F(v)$ is odd by assumption, so $|V|$ is even.

   Proposed by Bill Huang. Solution by Aleksa Milojević.

3. Let MN be a chord of a circle, and let S be its midpoint. Now let A, B, C, D be points on that circle such that AC and BD both contain S, and A and B are on the same side of MN. Let d_A, d_B, d_C, d_D be the distances from A, B, C, D respectively to MN. Prove that $\frac{1}{d_A} + \frac{1}{d_D} = \frac{1}{d_B} + \frac{1}{d_C}$.

   Solution: It's natural to convert the expression into $\frac{1}{d_A} - \frac{1}{d_C} = \frac{1}{d_B} - \frac{1}{d_D}$. Now we can see that since B, D can be any chord through S, we need to prove that $\frac{1}{d_A} - \frac{1}{d_C} = \textit{constant}$, where that constant only depends on MN and not on the choice of A. Now let's just compute it. Let O be the center of the circle and let the angle between AC and MN be θ. WLOG let $AS \leq CS$. Let the feet of perpendiculars from A and C to MN be A' and C'. Now from $\triangle AA'S \sim \triangle CC'S$ we have $\frac{d_A}{AS} = \frac{d_C}{CS}$, so now $\frac{1}{d_A} - \frac{1}{d_C} = \frac{1}{d_A}(1 - \frac{AS}{CS}) = \frac{1}{d_A}\frac{AS - CS}{CS}$. Let P be a point on AC such that $AS = CP$, now $\frac{1}{d_A} - \frac{1}{d_C} = \frac{1}{d_A}\frac{SP}{CS}$. Now we see that $\angle OSC = 90 - \theta$, so $SP = 2OS\sin\theta$. Now from the power of the point S, we have $AS \cdot CS = MS \cdot NS$, so since $d_A = AS\sin\theta$, we have that $\frac{1}{d_A} - \frac{1}{d_C} = \frac{1}{AS\sin\theta}\frac{2OS\sin\theta}{CS} = \frac{2OS}{MS \cdot NS}$, which doesn't depend on A. So now the result follows.

   Proposed by Igor Medvedev.