Combinatorics A Solutions

1. Prinstan Trollner and Dukejukem are competing at the game show WASS. Both players spin a wheel which chooses an integer from 1 to 50 uniformly at random, and this number becomes their score. Dukejukem then flips a weighted coin that lands heads with probability 3/5. If he flips heads, he adds 1 to his score. A player wins the game if their score is higher than the other player's score. The probability Dukejukem defeats the Trollner to win WASS equals m/n where m, n are coprime positive integers. Compute $m + n$.

   Proposed by Kapil Chandran.

   Answer: $\boxed{751}$.

   Solution: If the coin has probability q of landing heads, the probability of Dukejukem winning is $(1 - \mathbb{P}(\text{tie}))/2 + q\mathbb{P}(\text{tie})$, where $\mathbb{P}(\text{tie}) = 1/50$ is the probability that both players spin the same number on the wheel. This is 251/500, so the answer is 751.

2. Keith has 10 coins labeled 1 through 10, where the ith coin has weight 2^i. The coins are all fair, so the probability of flipping heads on any of the coins is $\frac{1}{2}$. After flipping all of the coins, Keith takes all of the coins which land heads and measures their total weight, W. If the probability that $137 \le W \le 1061$ is m/n for coprime positive integers m, n, determine $m + n$.

   Proposed by Alan Yan.

   Answer: $\boxed{743}$.

   Solution: We note that these weights form binary numbers, except the "1" is omitted. Thus the numbers that are generated are exactly the even numbers between 2 and 2046, inclusive. Thus the number of possibilities is the number of even numbers between 138 and 1060, inclusive, which is exactly 462. There are $2^{10} = 1024$ possible weights, which gives us a probability of $462/1024 = 231/512$ for an answer of 743.

3. Marko lives on the origin of the Cartesian plane. Every second, Marko moves 1 unit up with probability 2/9, 1 unit right with probability 2/9, 1 unit up and 1 unit right with probability 4/9, and he doesn't move with probability 1/9. After 2019 seconds, Marko ends up on the point (A, B). What is the expected value of $A \cdot B$?

   Proposed by Alan Yan.

   Answer: $\boxed{1811716}$.

   Solution: Define the random variables x_i, y_i for $1 \le i \le 2019$ where each x_i equals 1 if on the ith move, Marko makes a contribution to the right and zero otherwise. y_i is equal to 1 if on the ith move we make a contribution upwards and 0 otherwise. Hence, the answer is

$$\mathbb{E}\left[\sum_{i=1}^{n} x_i \cdot \sum_{j=1}^{n} y_j\right] = \sum_{i,j} \mathbb{E}[x_i y_j] = \frac{4n^2}{9} = 1811716.$$

   Note that one must do cases on whether $i = j$, but the numbers are such that everything is 4/9.

4. Kelvin and Quinn are collecting trading cards; there are 6 distinct cards that could appear in a pack. Each pack contains exactly one card, and each card is equally likely. Kelvin buys packs until he has at least one copy of every card, then he stops buying packs. If Quinn is missing exactly one card, the probability that Kelvin has at least two copies of the card Quinn is missing is expressible as m/n for coprime positive integers m, n. Determine $m + n$.

   Proposed by Sam Mathers.

Answer: $\boxed{191}$.

Solution: However, we also have the probabilities for each of the new cards that appear. This is $\frac{5}{6} \cdot \frac{4}{6} \cdot \dots \cdot \frac{1}{6} \cdot \frac{1}{6}$ since we are fixing when A appears so we have two copies of $\frac{1}{6}$, one for A and one for the last card that isn't A, Thus, in total, the probability is $\frac{6^5}{5! \cdot (6-n)} \cdot \frac{5!}{6^6} = \frac{1}{(6-n)6}$. We now need to sum this over all possible n, giving us

$$\sum_{n=0}^{5} \frac{1}{(6-n)6} = \frac{1}{6}(1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{6}) = \frac{49}{120}.$$

Since we computed the complement, the probability we want is $1 - \frac{49}{120} = \frac{71}{120}$.

5. A candy store has 100 pieces of candy to give away. When you get to the store, there are five people in front of you, numbered from 1 to 5. The ith person in line considers the set of positive integers congruent to i modulo 5 which are at most the number of pieces of candy remaining. If this set is empty, then they take no candy. Otherwise they pick an element of this set and take that many pieces of candy. For example, the first person in line will pick an integer from the set $\{1, 6, ..., 96\}$ and take that many pieces of candy. How many ways can the first five people take their share of candy so that after they are done there are at least 35 pieces of candy remaining?

   Proposed by Alan Chung.

   Answer: $\boxed{3003}$.

   Solution: We write the product of the generating functions for each person in line. The polynomial is as follows:

$$(x + x^6 + ...)(x^2 + x^7 + ...)...(x^5 + x^{10} + ...)$$

$$= x^{15}(1 + x^5 + x^{10} + ...)^5$$

   The coefficient of x^n in this polynomial represents the number of ways for the five people to take a total of n candies. The answer will be the sum of the coefficients of the terms of degree 15 thru 65. Thus, the answer is

$$\sum_{i=4}^{14} \binom{i}{4} = \binom{15}{5} = 3003.$$

6. The Nationwide Basketball Society (NBS) has 8001 teams, numbered 2000 through 10000. For each n, team n has $n+1$ players, and in a sheer coincidence, this year each player attempted n shots and on team n, exactly one player made 0 shots, one player made 1 shot, ..., one player made n shots. A player's *field goal percentage* is defined as the percentage of shots that the player made, rounded to the nearest tenth of a percent. (For instance, 32.45% rounds to 32.5%.) A player in the NBS is randomly selected among those whose field goal percentage is 66.6%. If this player plays for team k, what is the probability that $k \geq 6000$?

   Proposed by Zackary Stier.

   Answer: $\boxed{40007}$.

   Solution: We use Pick's theorem, $A = i + \frac{b}{2} - 1$ for A the area of an enclosed figure, i the number of interior lattice points, and b the number of boundary lattice points. We draw the triangle from the origin to the points $P = (6655, 10000)$ and $Q = (6665, 10000)$. The is of interest because it contains all points whose rise over run gives a fraction that we seek. We compute the number of lattice points N above the bottom edge in the trapezoid bounded by $(1331, 2000), P, Q, (1333, 2000)$, corresponding to the case of at least 2000. The bottom edge there has 5 lattice points $(1333, 2000)c$ for $c \in \{1, 2, 3, 4, 5\}$. We compute $N = 48006$, since there are $3 + 11 + 5 + 5 - 4 = 20$ boundary points. We similarly compute the number of lattice points M above the bottom edge in the trapezoid bounded by $(3993, 6000), P, Q, (3999, 6000)$ (corresponding to the case of at least 6000) as $M = 32008$, since there are $7+11+3+3-4 = 20$ boundary points. $\frac{m}{n} = \frac{M}{N} = \frac{32008}{48006} = \frac{16004}{24003}$, which is reduced, giving our answer to be 40007.

7. In the country of PUMACsboro, there are n distinct cities labelled 1 through n. There is a rail line going from city i to city j if and only if $i < j$; you can only take this rail line from city i to city j. What is the smallest possible value of n, such that if each rail line's track is painted orange or black, you can always take the train between 2019 cities on tracks that are all the same color? (This means there are some cities $c_1, c_2, \ldots, c_{2019}$, such that there is a rail line going from city c_i to c_{i+1} for all $1 \leq i \leq 2018$, and their rail lines' tracks are either all orange or all black.)

   Proposed by Reed Jacobs.

   Answer: $\boxed{4072325}$.

   Solution: We translate this into graph theory and solve a more generalized problem: Given a positive integer n, consider a complete directed graph $K_n^{\uparrow}$ whose vertices are n distinct real numbers. If $r < s$ are two vertices, direct edge $\{r, s\}$ to go out of r and into s. What is the smallest positive integer n, such that when the edges of $K_n^{\uparrow}$ are colored with c colors, a monochromatic directed path of length ℓ is guaranteed?

   For every vertex x and integer $1 \leq i \leq r$, let $a(x, i)$ be the number of edges in the longest monochromatic directed path of color i which ends at x. The problem is equivalent to showing that some $a(x, i)$ is at least $\ell - 1$; suppose this does not hold. Then there are at most $(\ell - 1)^c$ possible tuples $(a(x, 1), \ldots, a(x, a))$. Also, note that any two of these tuples are distinct; the definition of $K_n^{\uparrow}$ ensures the existence of a suitably directed edge, and the tuples differ at that edge's color. If we take $n = (\ell - 1)^c + 1$, we obtain a contradiction. Exhibiting a coloring of $K_{(\ell-1)^c}^{\uparrow}$ with c colors that has no monochromatic directed paths of length ℓ is left to the reader; as a hint, arrange the vertices in a c-dimensional hypercube. So the answer is $(2019 - 1)^2 + 1 = 4072325$.

8. Let S_n be the set of points $(x/2, y/2) \in \mathbb{R}^2$ such that x, y are odd integers and $|x| \leq y \leq 2n$. Let T_n be the number of graphs G with vertex set S_n satisfying the following conditions:

   - G has no cycles.
   - If two points share an edge, then the distance between them is 1.
   - For any path $P = (a, \ldots, b)$ in G, the smallest y-coordinate among the points in P is either that of a or that of b. However, multiple points may share this y-coordinate.

   Find the 100th-smallest positive integer n such that the units digit of T_{3n} is 4.

   Proposed by Michael Gintz.

   Answer: $\boxed{399}$.

   Solution: Note that S_n is just an upside-down pyramid. We wish to show that we can build G one row at a time from the biggest row. Note that if we do this, and we build a row, we have some partition of the row into segments, and each segment has at most one vertical line coming up from it. Note that then regardless of what lies above, if it follows the conditions in the problem then when we add these edges we will still follow the conditions in the problem. Now we can look at this inductively.

We wish to see how many ways $f(k)$ there are to build a row of length k (where k might be odd). Note that we must partition the row into segments, decide whether each segment gets a vertical line, and if it does where it goes. Then there are $x+1$ options for vertical lines for a segment of length x. Thus for $x \geq 1$ we can build this inductively based on the size of the first segment:

$$f(x) = 2f(x-1) + 3f(x-2) + \ldots + (x+1)f(0).$$

Thus we can add and subtract $f(x-1)$ from this so for $x \geq 2$ we have

$$f(x) = 3f(x-1) + f(x-2) + \ldots + f(0).$$

Finally we can add and subtract $f(x-1)$ again so for $x \geq 3$ we have

$$f(x) = 4f(x-1) - 2f(x-2).$$

We then have $f(0) = 1, f(1) = 2, f(2) = 7$ and we can build the rest from this last rule. Now note that

$$T_n = f(2) \times f(4) \times \ldots \times f(2n-2) \times 2^{2n-1},$$

where the 2^{2n-1} comes from the number of ways to partition the biggest row, with two choices for every adjacent pair. Let us list out the first few values of f mod 10. Starting from $f(0)$ we have

$$1, 2, 7, 4, 2, 0, 6, 4, 4, 8, 4, 0, 2, \ldots$$

Thus we see that $f(6+x) \equiv 2f(x) \pmod{10}$ for $x > 3$. Now we wish to take the product of the even ones. Note that for $x \geq 2$ we have

$$T_{x+3} \equiv 2^6 f(2x) f(2x+2) f(2x+4) T_x$$

Note that when x is 1,2,3 mod 3, the values of these products of f are equivalent to $2 \times 2^{x-4}$, $8 \times 2^{x-2}$ and $6 \times 2^{x-3}$. Now note that we can calculate the last digits of T_1 through T_4 as 2, 6, 8, 2. Thus we have for $x \geq 0$ we have

$$T_{3x+3} \equiv 8 \times 2^{6x} 6^x 2^{3x(x+1)/2} \equiv 8 \times 2^{6x} \times 2^{3x(x+1)/2}$$

We know that the second term cycles through 6,4 and the third cycles through 6,8,2,4,4,2,8,6 so T_{3x+3} cycles starting at $x = 0$ through 8,6,6,8,2,4,4,2. Thus each 8-cycle has 2 fours. Thus we want to be in the fiftieth cycle and take the seventh element which is $T_{3\times(8\times 49+6)}$ so our answer is $8 \times 49 + 7 = 399$.