Algebra A Solutions

1. Let x and y be positive real numbers that satisfy $(\log x)^2 + (\log y)^2 = \log(x^2) + \log(y^2)$. Compute the maximum possible value of $(\log xy)^2$.

   Proposed by: Matthew Kendall

   Answer: $\boxed{16}$

   Let $u = \log x$ and $v = \log y$. Then $u^2 + v^2 = 2u + 2v$. Completing the square gives $(u-1)^2 + (v-1)^2 = 2$, so the equation given is a circle of radius $\sqrt{2}$ centered at $(1, 1)$ on the uv plane.

   Let $\log xy = u + v = k$, so we wish to maximize k^2. Note that the line $u + v$ is tangent to the circle when the origin is a distance of 0 or $2\sqrt{2}$ from the line. The latter gives $u = v = 2$, so $k = 4$, making the maximum $k^2 = \boxed{16}$.

2. Let $f(x) = x^2 + 4x + 2$. Let r be the difference between the largest and smallest real solutions of the equation $f(f(f(f(x)))) = 0$. Then $r = a^{\frac{p}{q}}$ for some positive integers a, p, q so a is square-free and p, q are relatively prime positive integers. Compute $a + p + q$.

   Proposed by: Kevin Feng

   Answer: $\boxed{35}$

   Note that $f(x) = x^2 + 4x + 2 = (x+2)^2 - 2$. Then $f^2(x) = ((x+2)^2 - 2) + 2)^2 - 2 = (x+2)^4 - 2$. It is easy to see by induction that $f^n(x) = (x+2)^{2^n} - 2$, so $f^4(x) = (x+2)^{2^4} - 2$.

   Then the real solutions to $f^4(x) = 0$ are at $x + 2 = \pm \sqrt[16]{2}$, or $x = -2 \pm \sqrt[16]{2}$. Hence, the difference between the two of them are $2\sqrt[16]{2} = 2^{\frac{17}{16}}$, which gives us an answer of $2 + 16 + 17 = \boxed{35}$.

3. Let Q be a quadratic polynomial. If the sum of the roots of $Q^{100}(x)$ (where $Q^i(x)$ is defined by $Q^1(x) = Q(x), Q^i(x) = Q(Q^{i-1}(x))$ for integers $i \geq 2$) is 8 and the sum of the roots of Q is S, compute $|\log_2(S)|$.

   Proposed by: Matthew Kendall

   Answer: $\boxed{96}$

   Let the sum of the roots of $Q^j(x)$ be S_j for $j = 1, \ldots, 2019$. Our claim is $S_{j+1} = 2S_j$. Let $Q(x) = a(x-r)(x-s)$, where r and s are the roots of Q. Note that

   $$Q^{j+1}(x) = a(Q^j(x) - r)(Q^j(x) - s),$$

   so the solutions to $Q^{j+1}(x) = 0$ are the solutions to $Q^j(x) = r$ and $Q^j(x) = s$. Since the degree of Q^j is at least 1, the sum of the roots to $Q^j(x) = r$ and $Q^j(x) = s$ are both S_j, so $S_{j+1} = S_j + S_j = 2S_j$.

   From our recursion we get $S_{100} = 2^{99} S_1$. Therefore, $S_1 = \frac{8}{2^{99}}$ and $|\log_2(S)| = \boxed{96}$.

4. Let $\mathbb{N}_0$ be the set of non-negative integers. There is a triple (f, a, b), where f is a function from $\mathbb{N}_0$ to $\mathbb{N}_0$ and $a, b \in \mathbb{N}_0$, that satisfies the following conditions:

   1) $f(1) = 2$

   2) $f(a) + f(b) \leq 2\sqrt{f(a)}$

   3) For all $n > 0$, we have $f(n) = f(n-1)f(b) + 2n - f(b)$

   Find the sum of all possible values of $f(b + 100)$.

   Proposed by: Rahul Saha

Answer: $\boxed{10201}$

We'll focus on condition 2.

By AM-GM (or squaring and rearranging),

$$2\sqrt{f(a)f(b)} \leq f(a) + f(b) \leq 2\sqrt{f(a)}$$

which implies $f(b) \leq 1$. Since $f(b)$ is as integer we must have $f(b) = 0, 1$.

Substituting in condition (3) gives us the possibilities $f(n) = 2n$ for $n > 0$ (for $f(b) = 0$) and a recursion which easily amounts to $f(n) = n^2 + 1$.

For the first function, since $f(n) = 2n$ for $n > 0$ and $f(b) = 0$, we must necessarily have $b = 0$. So $f(b + 100) = f(100) = 200$.

In the second case, similarly $b = 0$ and $f(b + 100) = 100^2 + 1 = 10001$.

Summing gives us the answer $\boxed{10201}$.

Quick check: In both cases, if we have $a = b = 0$, condition 2 holds. Condition 1 works for both functions too. So our functions do satisfy the problem's statement.

5. Let $\omega = e^{\frac{2\pi i}{2017}}$ and $\zeta = e^{\frac{2\pi i}{2019}}$. Let $S = \{(a, b) \in \mathbb{Z} \mid 0 \leq a \leq 2016, 0 \leq b \leq 2018, (a, b) \neq (0, 0)\}$. Compute $\prod_{(a,b)\in S} (\omega^a - \zeta^b)$.

   Proposed by: Frank Lu

   Answer: $\boxed{4072323}$

   First, fix a. Note that $\prod_{b=0}^{2018} (x - \zeta^b) = x^{2019} - 1$. Hence, if $a \neq 0$, $\prod_{b=0}^{2018} (\omega^a - \zeta^b) = \omega^{2019a} - 1$. For $a = 0$, we have that this is $\prod_{b=1}^{2018} (1 - \zeta^b) = 2019$, since $\prod_{b=1}^{2018} (x - \zeta^b) = \prod_{b=0}^{2018} (x - \zeta^b)/(x - 1) = \sum_{b=0}^{2018} x^b$. Thus, our product becomes $\prod_{a=1}^{2016} (\omega^{2019a} - 1) * 2019$. But note that this then becomes $2017 * 2019$, since the ω^{2019a} are just a permutation of the 2017th roots of unity besides 1 (as 2017 and 2019 are relatively prime), which is then just $\boxed{4072323}$.

6. A *weak binary representation* of a nonnegative integer n is a representation $n = a_0 + 2 \cdot a_1 + 2^2 \cdot a_2 + \ldots$ such that $a_i \in \{0, 1, 2, 3, 4, 5\}$. Determine the number of such representations for 513.

   Proposed by: Frank Lu

   Answer: $\boxed{3290}$

   Let $N(k)$ be the number of such representations for k. We know that $N(0) = 1, N(1) = 1, N(2) = 2, N(3) = 2$, and $N(4) = 4$. We can see, based on the choice of a_0, that $N(2k) = N(2k+1) = N(k) + N(k-1) + N(k-2)$. To make use of this recurrence relation, we define two sequences. First, define $x_k = N(2k)$. Observe then that $x_{2k} - x_{2k-1} = N(4k) - N(4k - 2) = N(2k) - N(2k - 3) = x_k - x_{k-2}$, and that $x_{2k+1} - x_{2k} = x_k - x_{k-1}$ by a similar token. Now, let $y_k = x_k - x_{k-1}$. Then, our recurrence relation becomes $y_{2k+1} = y_k$ and that $y_{2k} = y_k + y_{k-1}$. From our earlier cases before we see that $y_1 = 1$ and $y_2 = 2$. Based on the recurrence in the odd case, we see that $y_{2^k-1} = 1$ for each integer i.

   Claim: $\sum_{i=2^{k-1}-1}^{2^k-2} y_i = 3^{k-1}$.

Proof: We inductively show this. For $k = 2$ we can easily verify this. Now, given the k case, note that $\sum_{i=2^k-1}^{2^{k+1}-2} y_i = \sum_{i=2^{k-1}-1}^{2^k-2} y_{2i+1} + \sum_{i=2^{k-1}}^{2^k-1} y_{2i}$. Using our recurrence, this becomes $\sum_{i=2^{k-1}-1}^{2^k-1} y_i + \sum_{i=2^{k-1}}^{2^k-1} y_i + \sum_{i=2^{k-1}}^{2^k-1} y_{i-1}$. But knowing that $y_{2^k-1} = y_{2^{k-1}-1} = 1$ yields that this is just $3 * \sum_{i=2^{k-1}-1}^{2^k-2} y_i = 3^{k-1} * 3 = 3^k$, proving the inductive case and proving the claim. Finally, observe that $N(513) = N(512) = x_{256} = x_0 + \sum_{i=1}^{256} y_i$, which using our claim is just $1 + 3 + 3^2 + \ldots + 3^7 + y_{255} + y_{256}$. A final observation that $y_{2*k} = k + 1$ yields the answer $(3^8 - 1)/2 + 1 + 9 = \boxed{3290}$.

7. A doubly-indexed sequence $a_{m,n}$, for m and n nonnegative integers, is defined as follows.

   (a) $a_{m,0} = 0$ for all $m > 0$ and $a_{0,0} = 1$.

   (b) $a_{m,1} = 0$ for all $m > 1$, and $a_{1,1} = 1, a_{0,1} = 0$.

   (c) $a_{0,n} = a_{0,n-1} + a_{0,n-2}$ for all $n \geq 2$

   (d) $a_{m,n} = a_{m,n-1} + a_{m,n-2} + a_{m-1,n-1} - a_{m-1,n-2}$ for all $m > 0, n \geq 2$.

   Then there exists a unique value of x so $\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{a_{m,n} x^m}{3^{n-m}} = 1$. Find $\lfloor 1000x^2 \rfloor$.

   Proposed by: Frank Lu

   Answer: $\boxed{27}$

   Define the sequence of polynomials $P_n(x)$ by $P_0(x) = 1, P_1(x) = x$, and for $n \geq 2$ $P_n(x) = (x+1)P_{n-1}(x) - (x-1)P_{n-2}(x)$. Observe that our given sequence is uniquely determined by the values when $n = 0$ and $n = 1$, over all m. Letting $b_{m,n}$ be the coefficient of x^m in $P_n(x)$, our recurrence becomes $b_{m,n} = b_{m,n-1} + b_{m,n-2} + b_{m-1,n-1} - b_{m-1,n-2}$, and that the $b_{m,n}$ satisfy the same initial conditions that are given. It thus follows that our b sequence is the exact same as the given a sequence. Now, define the function $g(x, y) = \sum_{n=0}^{\infty} P_n(x)y^n$. Observe that, based on our recurrence and the initial conditions, we have that $g(x, y) = 1 + xy + (x+1)y(g(x, y) - 1) - (x-1)y^2(g(x, y))$. Rearranging this and solving for g gives us now that $g(x, y) = \frac{1-y}{y^2(x-1)-y(x+1)+1}$. But our desired sum can be written as $g(3x, 1/3)$. Let $u = 3x$. Thus, our desired u is so that $\frac{\frac{2}{3}}{\frac{1}{9}(u-1)-\frac{1}{3}(u+1)+1} = 1$, or that $\frac{2}{3} = \frac{-2u}{9} + \frac{5}{9}$, or $u = \frac{-1}{2}$, which means that $x = \frac{-1}{6}$. This gives us an answer of $\lfloor \frac{1000}{36} \rfloor = \lfloor \frac{250}{9} \rfloor = \boxed{27}$

8. For real numbers a and b, define the sequence $\{x_{a,b}(n)\}$ as follows: $x_{a,b}(1) = a$, $x_{a,b}(2) = b$, and for $n > 1$, $x_{a,b}(n+1) = (x_{a,b}(n-1))^2 + (x_{a,b}(n))^2$. For real numbers c and d, define the sequence $\{y_{c,d}(n)\}$ as follows: $y_{c,d}(1) = c$, $y_{c,d}(2) = d$, and for $n > 1$, $y_{c,d}(n+1) = (y_{c,d}(n-1) + y_{c,d}(n))^2$. Call (a, b, c) a good triple if there exists d such that for all n sufficiently large, $y_{c,d}(n) = (x_{a,b}(n))^2$. For some (a, b) there are exactly three values of c that make (a, b, c) a good triple. Among these pairs (a, b), compute the maximum value of $\lfloor 100(a + b) \rfloor$.

   Proposed by: Eric Neyman

   Answer: $\boxed{120}$

   Define (a, b, c, d) to be *good* if for n large enough, $y_{c,d}(n) = (x_{a,b}(n))^2$. Fix a good quadruple (a, b, c, d). For brevity of notation, we will denote $x_{a,b}(n)$ as x_n and $y_{c,d}(n)$ as y_n.

We claim that $y_n = x_n 2$ for all $n \geq 3$. Suppose for contradiction that this is not the case, and let $k \geq 3$ be such that $y_n = x_n^2$ for all $n > k$, but $y_k \neq x_k^2$. We have

$$\begin{aligned} y_{k+2} &= x_{k+2}^2 \\ (y_k + y_{k+1})^2 &= (x_k^2 + x_{k+1}^2)^2 \\ y_k + y_{k+1} &= \pm(x_k^2 + x_{k+1}^2) \\ y_k + y_{k+1} &= \pm(x_k^2 + y_{k+1}). \end{aligned}$$

We can't choose the plus sign because then $y_k = x_k^2$, which we assumed to not be the case. Thus, $y_k + y_{k+1} = -x_k^2 - y_{k+1}$, so

$$y_k = -x_k^2 - 2y_{k+1} = -x_k^2 - 2(y_{k-1} + y_k)^2 \leq 0.$$

But $y_k = (y_{k-2} + y_{k-1})^2 \geq 0$, so $y_k = 0$. This means that $x_k^2 = 0$, so $x_k = 0$, contradicting our assumption that $y_k \neq x_k^2$. Therefore, $y_n = x_n^2$ for all $n \geq 3$.

Suppose that (a, b, c, d) is good. We have

$$x_1 = a, x_2 = b, x_3 = a^2 + b^2, x_4 = b^2 + (a^2 + b^2)^2$$

and

$$y_1 = c, y_2 = d, y_3 = (c + d)^2, y_4 = (d + (c + d)^2)^2.$$

Since $y_3 = x_3^2$ and $y_4 = x_4^2$, we have the equations

$$c + d = \pm(a^2 + b^2) \tag{1}$$

and

$$d + (c + d)^2 = \pm(b^2 + (a^2 + b^2)^2). \tag{2}$$

Plugging in $(a^2 + b^2)^2$ for $(c + d)^2$ in (2), we have

$$d + (a^2 + b^2)^2 = \pm(b^2 + (a^2 + b^2)^2).$$

This gives two possibilities: $d = b^2$ or $d = -b^2 - 2(a^2 + b^2)^2$.

Suppose that $d = b^2$. Then (1) gives $c + b^2 = \pm(a^2 + b^2)$, so c is either a^2 or $-a^2 - 2b^2$.

Suppose that $d = -b^2 - 2(a^2 + b^2)^2$. Then (2) gives

$$c - b^2 - 2(a^2 + b^2)^2 = \pm(a^2 + b^2),$$

so c is either $2(a^2 + b^2)^2 - a^2$ or $a^2 + 2b^2 + 2(a^2 + b^2)^2$.

Note that all four of the values of c that are listed work, because all our steps can be reversed and if $x_k^2 = y_k$ and $x_{k+1}^2 = y_{k+1}$, then $x_n^2 = y_n$ for all $n \geq k$.

We want exactly two of the four listed values of c to be equal. Note that if $a = 0$ then the four values of c are 0, $-2b^2$, $2b^4$, and $2b^2 + 2b^4$, which are all different unless $b = 0$, in which case they are all the same. Thus, we may assume that $a \neq 0$. This means that $2(a^2 + b^2)^2 - a^2 < a^2 + 2b^2 + 2(a^2 + b^2)^2$, $a^2 < a^2 + 2b^2 + 2(a^2 + b^2)^2$, $-a^2 - 2b^2 < a^2$, and $-a^2 - 2b^2 < 2(a^2 + b^2)^2 - a^2$. Thus, for two of the values of c to be the same, we must have $2(a^2 + b^2)^2 - a^2 = a^2$, i.e. $(a^2 + b^2)^2 = a^2$. Thus, $a^2 + b^2 = \pm a$, so $\left(a \pm \frac{1}{2}\right)^2 + b^2 = \frac{1}{4}$. This means that (a, b) is a point on either the circle with radius $\frac{1}{2}$ centered at $\left(\frac{1}{2}, 0\right)$ or the circle with radius $\frac{1}{2}$ centered at $\left(\frac{-1}{2}, 0\right)$. $a + b$ is maximized at the point where the rightmost circle is tangent to a line with slope -1 that is "furthest right." This happens at the point $\left(\frac{1}{2} + \frac{\sqrt{2}}{4}, \frac{\sqrt{2}}{4}\right)$, where $a + b = \frac{1}{2} + \frac{\sqrt{2}}{2} = \frac{1+\sqrt{2}}{2}$.

Thus, our answer is $\lfloor 50 + 50\sqrt{2} \rfloor = 50 + 70 = \boxed{120}$.